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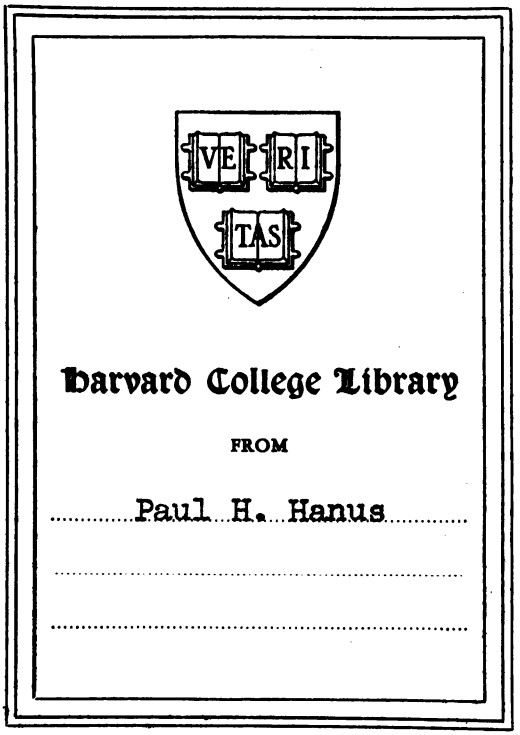

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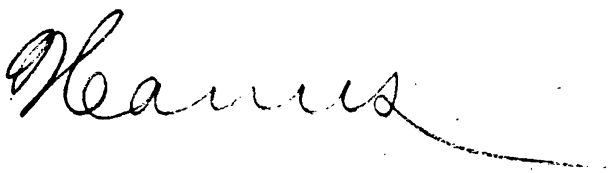

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KEY TO SCIENCE OF ARITHMETIC.

NOTATION.

PAGES **12** AND **13.**

(1.) $5am = 5 \times 3 \times 7 = 105$ $5am = 5 \times 100 \times 8 = 4000$.

(2.) $32bcy = 32 \times 5 \times 50 \times 121 = 968000$ $32bcy = 32 \times 211 \times 27 \times 1 = 182304$.

(3.) $xy = 11 \times 125 = 1375$: $xy = 1 \times 1 = 1$ $xy = 10 \times 10 = 100$: $xy = 3 \times 3 \times 3 = 9$.

(4.) $b \times a = ba$, or ab, which are equivalent.

(5.) If 1 load contain x bushels, m loads contain m times x bushels, or mx bushels.

(6.) b cents $\times$ $a = ab$ cents; and ab cents $\times$ by $x = abx$.

(7.) a times m bricks are am bricks; and 26 times am bricks are $26am$ bricks.

REDUCTION.

PAGES **21–23.**

(2.)

1302_5	10200_5
5	5
8	5
5	5
40	27
5	5
202_{10}	135
	5
	675_{10}

(3.)

40210_5	1111_5
5	5
20	6
5	5
102	31
5	5
511	156_{10}
5	
2555_{10}	

(4.)

10_5 | 100_5 | 1000_5

$\begin{array}{r} 5 \\ \hline 5_{10} \end{array}$ $\quad$ $\begin{array}{r} 5 \\ 5 \\ \hline 25_{10} \end{array}$ $\quad$ $\begin{array}{r} 5 \\ 5 \\ 5 \\ 25 \\ \hline 125_{10} \end{array}$

(6.)

$\begin{array}{r|l} 5)68-3 & \\ 5)13-3 & \\ \hline 2 & \end{array}$ $\quad$ $\begin{array}{r|l} 5)111-1 & \\ 5)22-2 & \\ \hline 4 & \end{array}$ $\quad$ $\begin{array}{r|l} 5)20-0 & \\ \hline 4 & \end{array}$ $\quad$ $\begin{array}{r|l} 5)10-0 & \\ \hline 2 & \end{array}$

$\therefore 68_{10}=233_5.$ $\quad$ $\therefore 111_{10}=421_5.$ $\quad$ $\therefore 20_{10}=40_5.$ $\quad$ $\therefore 10_{10}=20_5.$

$\begin{array}{r|l} 5)3802-2 & \\ 5)760-0 & \\ 5)152-2 & \\ 5)30-0 & \\ 5)6-1 & \\ \hline 1 & \end{array}$ $\quad$ $\begin{array}{r|l} 5)100-0 & \\ 5)20-0 & \\ \hline 4 & \end{array}$

$\therefore 3802_5 = 110202_5.$ $\quad$ $\therefore 100_{10} = 400_5.$

(7.)

$\begin{array}{r|l} 7)6432-6 & \\ 7)918-1 & \\ 7)131-5 & \\ 7)18-4 & \\ \hline 2 & \end{array}$ $\quad$ $\begin{array}{r} 24516_7 \\ 7 \\ \hline 18 \\ 7 \\ \hline 131 \\ 7 \\ \hline 918 \\ 7 \\ \hline 6432_{10} \end{array}$ $\quad$ $\begin{array}{r|l} 6)4503-3 & \\ 6)750-0 & \\ 6)125-5 & \\ 6)20-2 & \\ \hline 3 & \end{array}$ $\quad$ $\begin{array}{r} 32503_6 \\ 6 \\ \hline 20 \\ 6 \\ \hline 125 \\ 6 \\ \hline 750 \\ 6 \\ \hline 4503_{10} \end{array}$

$9)800-8$
$9)88-7$
$9)9-0$
1

1078_9
9
9
9
88
9
800_{10}

201_4
4
8
4
33_{10}

$4)33-1$
$4)8-0$
2

$4)2343-3$
$4)585-1$
$4)146-2$
$4)36-0$
$4)9-1$
2

210213_4
4
9
4
36
4
146
4
585
4
2343_{10}

1000_3
3
3
3
9
3
27_{10}

$3)27-0$
$3)9-0$
$1)3-0$
1
$\therefore 1000_3 = 27_{10}.$

(8.)

$3\psi 0\theta 2_{12}$
12
47
12
564
12
6778
12
81338_{10}

(10.)

$4)1000_6-0$
$4)130-2$
$4)21-1$
3
$\therefore 1000_6 = 3120_4.$

$5)3120_4-1$
$5)223-3$
$5)20-3$
1
$\therefore 3120_5 = 1331_4.$

(59.)

RULE.—*To reduce a number represented in any scale to the equivalent number in the Decimal scale.*—Multiply the highest order in the given number by the radix of the

given scale, and to the product add the number in the next lower order. Multiply this sum by the same radix, and to the product add the next lower order. Proceed in this manner till the lowest order in the given number is reached.

NOTE.—This is the ordinary process of Reduction Descending.

RULE.—*To reduce a number represented in the Decimal scale to an equivalent number in any other scale.*—Divide the given number by the radix of the required scale, and reserve the remainder. Divide this quotient by the same radix and reserve the remainder. Proceed in this manner till a quotient is reached which is less than the radix of the required scale. This last quotient with the several remainders from the last to the first, written in order from left to right, will represent the number in the required scale.

NOTE.—1. This is the ordinary process of Reduction Ascending.

2. The latter of these rules will solve all cases if properly applied. Thus to reduce 324_8 to the decimal scale by division, we divide by 10. As 10 is not contained in 3, we reduce the 3 25s to 5s, making with the 2 5s, 17 5s. In this 10 is contained 1 5 times and 7 5s remainder. 7 5s are 35 units, which with the 4 units, make 39 units. In this 10 is contained 3 times with 9 remainder, etc.

$$\begin{array}{r|l} 10 & 324 - 9 \\ \hline 10 & 13 - 8 \\ \hline & 0 \end{array}$$

REDUCTION OF FRACTIONS.

PAGES 23–28.

(1–18.) The *operation* in the case of these examples is too familiar to need explanation. The purpose for which they are given in the text-book is that the *reasoning* as there elaborated may be made familiar, and especially that each may be seen to be a case in ordinary reduction, either ascending or descending.

(19.) $\frac{4}{21} = 4 \div 21 = .\dot{1}90476$:
$\frac{2}{33} = 2 \div 33 = .0\dot{6}$:
$\frac{7}{51} = 7 \div 51 = .\dot{1}372549019607843\dot{1}$
$\frac{1}{7} = 1 \div 7 = .\dot{1}42857\dot{1}$
$\frac{3}{7} = 3 \div 7 = .\dot{4}28571.$
$\frac{12}{17} = 12 \div 17 = .\dot{7}05882352941176\dot{4}.$

(21, 22.) These problems are the ordinary ones in reduction of decimals to common fractions, and the pupil is presumed to be familiar with the *operation*. The *explanation* is what we now seek to render familiar; the main purpose being to make it evident that every operation in reduction of fractions conforms either to the case of Reduction Descending or Reduction Ascending. Thus to reduce .48 to a common fraction we may explain that writing it in the form $\frac{48}{100}$ is but expressing the same thing in a different way. Then since 4 100ths make a 25th, there are as many 25ths in 48 100ths as 4 is contained times in 48, *i. e.*, 12; hence $\frac{48}{100} = \frac{12}{25}$. This case is thus seen to be the ordinary case of Reduction Ascending, in which the operation is performed by division. It is exactly analogous to the reduction of 48 *far.* to 12 *pence.*

(24.) Since $.\dot{1} = \frac{1}{9}$, and $.\dot{2}$ is 2 times $.\dot{1}$, $.\dot{2}$ is 2 times $\frac{1}{9}$, or $\frac{2}{9}$; etc.

(25.) That $.0\dot{1}$ is equivalent to $\frac{1}{90}$ is evident, since it is $\frac{1}{10}$ of $.\dot{1}$, as placing a 0 between the decimal point and the fraction divides the fraction by 10. Then, as $.0\dot{7}$ is 7 times $.0\dot{1}$, $.0\dot{7} = \frac{7}{90}$; etc.

(26.) As above, $.00\dot{1}$ is $\frac{1}{10}$ of $.0\dot{1}$, or $\frac{1}{100}$ of $.\dot{1}$, and hence $.00\dot{1} = \frac{1}{900}$. Then as $.00\dot{6}$ is 6 times $.00\dot{1}$, $.00\dot{6} = \frac{6}{900} = \frac{1}{150}$, etc.

(27.) $.0\dot{1} = .010101$, etc., to infinity. (This should be carefully distinguished from $.0\dot{1}$, which is .01111, etc., to infinity.) $.0\dot{1} = \frac{1}{99}$ as is found when we attempt to reduce

$\frac{1}{99}$ to a decimal. Then, as $.\dot{0}\dot{2}$ is 2 times $.\dot{0}\dot{1}$, $.\dot{0}\dot{2} = \frac{2}{99}$. In like manner $.\dot{0}\dot{6} = \frac{6}{99} = \frac{2}{33}$, etc.

(28.) $.\dot{0}0\dot{1}$ multiplied by 235 gives $.\dot{2}\dot{3}\dot{5}$, (.001001001, etc., to infinity × 235 gives .235235235, etc., to infinity.) Hence, $.\dot{2}\dot{3}\dot{5} = 235$ times $\frac{1}{999}$, or $\frac{235}{999}$, etc.

(29.) $.0\dot{2}\dot{7}$ is $\frac{1}{10}$ of $.\dot{2}\dot{7}$. But, as before shown, $.\dot{2}\dot{7}$ is $\frac{27}{99}$; hence, $.0\dot{2}\dot{7}$ is $\frac{1}{10}$ of $\frac{27}{99}$, or $\frac{27}{990} = \frac{3}{110}$. In like manner $.00\dot{2}\dot{7}$ is $\frac{1}{100}$ of $\frac{27}{99}$, or $\frac{27}{9900} = \frac{3}{1100}$, etc.

(30.) $.32\dot{6} = .32 + .00\dot{6} = \frac{32}{100} + \frac{6}{900} = \frac{294}{900} = \frac{49}{150}$.

$.53\dot{1}\dot{2} = .53 + .001\dot{2} = \frac{53}{100} + \frac{12}{9900} = \frac{5253}{9900} = \frac{1751}{3300}$.

$.62\dot{4}\dot{5} = .6 + .02\dot{4}\dot{5} = \frac{6}{10} + \frac{245}{9900} = \frac{6239}{9900}$.

$.301526\dot{6} = .301 + .000526\dot{6} = \frac{301}{1000} + \frac{526}{9999000} = \frac{13042}{330330}$.

$.3\dot{7} = .3 + .0\dot{7} = \frac{3}{10} + \frac{7}{90} = \frac{3 \times 9}{90} + \frac{7}{90} = \frac{3(10-1)+7}{90}$
$= \frac{30-3+7}{90} = \frac{37-3}{90} = \frac{34}{90} = \frac{17}{45}$.*

$.4036\dot{4}1\dot{2} = \frac{403}{1000} + \frac{6412}{9999000} = \frac{403(9999)+6412}{9999000} =$
$\frac{403(10000-1)+6412}{9999000} = \frac{4030000-403+6412}{9999000} =$
$\frac{4036412-403}{9999000} = \frac{4036009}{9999000}$.*

$.0028\dot{1}02\dot{1} = \frac{281021-28}{99990000} = \frac{280993}{99990000}$.

$.612\dot{4} = \frac{612}{1000} + \frac{4}{9000} = \frac{612 \times 9 + 4}{9000} = \frac{612(10-1)+4}{9000} =$
$\frac{6120-612+4}{9000} = \frac{6120+4-612}{9000} = \frac{6124-612}{9000} = \frac{5512}{9000}$
$= \frac{689}{1125}$.

(66.) RULE.—*To reduce a Pure Repetend to a common fraction.*—Write for the numerator the number represented by the figures of a period of the repetend, and for the denominator as many 9s as there are figures in the period.

To reduce a Mixed Repetend to a common fraction.—Write the part preceding the repeating period as a common

* From these two operations the following rule may be deduced for mixed repetends: From the number represented by the figures of the decimal, including the first period of the repetend, subtract the number represented by the figures preceding the repetend. The remainder is the numerator of the required fraction. The denominator consists of as many 9s as there are figures in the period, followed by as many 0s as precede the repeating figures.

fraction in the ordinary way. Write a period of the repetend as the numerator of a fraction, and for the denominator as many 9s as there are figures in the period, followed by as many 0s as there are figures preceding the first period in the given repetend. Then add the two portions thus obtained.

REDUCTION OF DENOMINATE NUMBERS.

PAGES 29–31.

(2.) $16 \times 3 = 48$. (3.) $16 \times .75 = 12$.

In analyzing such problems we may say, "Since there are 16 *oz.* in 1 *lb.*, in 3 *lbs.* there are 3 times 16 *oz.*, or 48 *oz.*" Or, we may say, "Since in every pound there are 16 *oz.*, in any number of pounds there are 16 times as many ounces as pounds; hence in 3 *lb.* there are 16 times 3, or 48 *oz.*" The former analysis has the advantage of simplicity, and would be that naturally presented to the beginner. The latter has the advantage of comprehensiveness and directness, since we usually use the radix as the multiplier. Observe also that it applies more elegantly to the case of fractions.

(4.) 16 times $3\frac{3}{4} = 60$.

(6.)
£4 10*s.* 7*d.*
20
90*s.*
12
1087*d.*

(7.) $\frac{3}{4} \times 12 \times 4 = 36$.

(8.) $\frac{1}{16} \times 12 \times 4 = \frac{3}{4}$.

(9.) $\frac{12}{12} \times \frac{6}{6} \frac{7}{8} \frac{2}{8} \frac{0}{0} = 10560$.

(12.) $\frac{12 \times 6280}{39.37} = 1609.34+$.

(13.) $\frac{20 \times 7000}{15432} = 9.072+$.

NOTE.—It will be best for the student to use only the fundamental numbers for comparison of our weights and measures with the *Metric*. Thus, 39.37 *in.* = 1 *meter*; 15.432 *gr.* Troy = 1 *gram*; .908 *dry quarts*, or 1.0567 *liquid quarts* = 1 *liter*; and that an *are* is 100 *sq. meters*; and a stere 1 *cu. meter*. A little effort will enable

him *to fix these numbers* in memory, when he will have command of the whole; whereas, if he attempt to remember many other relations he will soon forget all. Of course, if he were to be required to do a great deal of work requiring such reductions, he would need to use more expeditious numbers.

(14.) $\frac{640 \times 160 \times 272.25 \times 144}{39.37 \times 39.37 \times 100} = 25899.6+.$

(15.) $1.6 \div 12 = \frac{1.6}{12} = \frac{16}{120} = \frac{2}{15}.$

(16.) $\frac{46850 \times 1728}{231 \times 81.5} = 11125.79-.$

(17.) $\frac{1 \times 5280 \times 12}{39370} = 1.609+.$

(18.) $\frac{1 \times 4}{1.0567} = 3.785+.$

(19.) 1 stere being a cubic meter is $39.37 \times 39.37 \times 39.37$ *cu. in.* Hence we have

$$\frac{39.37 \times 39.37 \times 39.37}{128 \times 1728} = .276.$$

(20.)
$$\begin{array}{r|l} 30 & 25 \\ \hline 12 & 7.8333\dot{3} \\ \hline & .6527\dot{7} \end{array}$$

(21.) $\frac{1}{250} \times 12 = \frac{3}{250}.$

(22.) $\frac{1}{15} \times 365 = 24\frac{1}{3}.$

(23.) $\frac{39.37 \times 39.37 \times 39.37 \times 10}{128 \times 1728} = 2.76+.$ (See 19.)

(24.) $\frac{\frac{2}{3}}{3 \times \frac{1}{4} \times 51\frac{1}{4}} = \frac{1}{3 \times 4 \times 51\frac{1}{4}} = \frac{1}{615}.$

(25.) $\frac{.05 \times 12}{.03937} = \frac{6}{39.37} = 15.24.$

(26.) $\frac{24 \times 39.37}{12 \times 3} = 371.82+.$

(27.) $10 \times 2150.4 = 21504.$

(28.) $\frac{1 \times 16\frac{1}{2} \times 12}{39.37} = 5.03-.$

(29.) $\frac{100 \times 39.37 \times 39.37}{160 \times 272.25 \times 144} = 2.471+.$

NOTE.—It is exceedingly desirable that the pupil be taught to fix in mind the fundamental numbers as given under Ex. 12, and to obtain all results from them, *without consulting the tables of comparison.*

(30.) $\frac{100 \times .908}{32} = \frac{90.8}{32} = 2.8375.$

(31.)
$$\begin{array}{r|l} 2 & 1. \\ \hline 4 & 3.5 \\ \hline & .875. \end{array}$$

(32.) $3450 \div 19.3 = 178.75+.$

(33.) $8145 \div 23.8 = 342.23.$

(34.) $.193 \times 3000 = 579.$

(35.) $1728 \times 7 + 112 = 12208.$

(36.) $.03937 \times 42 = 1.65354.$

(37.) $\frac{21 \times 7000}{5760} = 3\frac{11}{288}.$

(38.) $3.937 \times 642 = 2527.554.$

(39.) $\frac{9}{9 \times \frac{486252}{801} \times 160} = 11.16.$

(40.)

.91225
12
10.94700
20
18.940
24
22.56

(41.)

$5.62542
100
562.542 *c.*
10
5625.42 *m.*

(42.) $\frac{7}{15} \times 320 = 133\frac{1}{3}.$ $\frac{1}{3} \times 16\frac{1}{2} = \frac{1}{3} \times \frac{33}{2} = 5\frac{1}{2}.$ $\frac{1}{2} \times 12 = 6.$ $\therefore \frac{7}{15}$ *mi.* $= 133$ *rd.* 5 *ft.* 6 *in.*

(43.) $\frac{7}{8} \times 128 = 112.$ (44.) $\frac{2}{5} \div 320 = \frac{1}{480}.$

(45.) $2\frac{3}{4} = \frac{11}{4}.$ $\frac{11}{4} \times 31\frac{1}{2} \times 231 \div 2150.4 = \frac{11 \times 63 \times 231}{2150.4} = 9.3.$

(46.) $\frac{6 \times 8 \times 3 \times 20}{1000} \times 16 = 6.58.$

(47.) $\frac{.015432 \times 25}{1000} \times 16 = .00088+.$

(48.) $24 \div 15.432 = 1.555+.$

(49.) $.03937 \times 250 = 9.8425.$

(50.)

3° 15′ 20″
60
195′
60
11720″

(51.)

60 | 42″
60 | 18.7′
| 3.31$\frac{1}{4}$.°

(52.) $1000 \div .193 = 5181.34;$ $1000 \div .238 = 4201.68.$

(53.) $\frac{5}{9}(68 - 32) = \frac{5 \times 36}{9} = 20.$

68° *Fah.* is 36° above freezing; and as a degree *Fah.* is $\frac{5}{9}$ of a degree C, 36° *Fah.* are 36 times $\frac{5}{9}$° C.

(54.) $22 \times \frac{9}{5} + 32 = 71\frac{3}{5}$.
(55.) $\frac{5}{9}(98 - 32) = 36\frac{8}{9}$.
(56.) $72 \times \frac{9}{5} + 32 = 161\frac{3}{5}$.

ADDITION.

PAGES 35–37.

(3.) This and kindred examples here are given simply to show that the principles involved in Simple and Compound Addition (and, in fact, in all addition) are identical.

(4.)
2472_8
5640_8
372_8
12_8
713_8
4_8
3111_8
6023_8
23011_8

There are two methods of looking at this solution. 1st. We may add each column in the ordinary manner, and then, since 8 of the particular order added make 1 of the next, divide the sum by 8, write the remainder under the column added, and add the quotient to the next column. 2d. Or in adding if we recollect how numbers are represented in the *octary* scale, the process will be exactly like the common process of simple addition in the decimal scale; that is, we shall write down the right hand figure of each sum and "carry" the left. Thus, adding the units column in this way, the sums are 3, 4, 10,* 13, 15, 17, 21†: writing the 1 under units' column, we add the 2 8s to the next column. Adding the 8's column we have 4, 5, 6, 7, 16, 22, 31: writing the 1 8 under 8's column, we add the 3 64s to the 3d column which is 64s. The sums in adding this column are 4, 13, 16, 24, 30, etc.

NOTE.—Observe that in whatever scale numbers are written, *if our Addition and Multiplication Tables are conformed to that scale, the fundamental operations* (addition, subtraction, multiplication, and division,) *will be performed in the ordinary manner now used in the decimal scale.*

* This is the way that 8 is represented in the octary scale.
† This is 2 8's and 1.

We will give the

OCTARY ADDITION TABLE.

1+1 = 2	2+2 = 4	3+3 = 6	4+4 = 10	5+5 = 12
1+2 = 3	2+3 = 5	3+4 = 7	4+5 = 11	5+6 = 13
1+3 = 4	2+4 = 6	3+5 = 10	4+6 = 12	5+7 = 14
1+4 = 5	2+5 = 7	3+6 = 11	4+7 = 13	6+6 = 14
1+5 = 6	2+6 = 10	3+7 = 12		6+7 = 15
1+6 = 7	2+7 = 11			7+7 = 16
1+7 = 10				

and also the

OCTARY MULTIPLICATION TABLE.

1×1 = 1	2×2 = 4	3×3 = 11	4×4 = 20	5×5 = 31
1×2 = 2	2×3 = 6	3×4 = 14	4×5 = 24	5×6 = 36
1×3 = 3	2×4 = 10	3×5 = 17	4×6 = 30	5×7 = 43
1×4 = 4	2×5 = 12	3×6 = 22	4×7 = 34	6×6 = 44
1×5 = 5	2×6 = 14	3×7 = 25		6×7 = 52
1×6 = 6	2×7 = 16			7×7 = 61
1×7 = 7				

(8.) Because these numbers represent things of totally different kinds, and hence can not be reduced to a common denomination.

(9.)

36 *gr.*	=	1 *pwt.* 12 *gr.*
81 *pwt.*	=	4 *oz.* 1 *pwt.*
30 *oz.*	=	2 *lb.* 6 *oz.*
2 *lb.*	=	2 *lb.*
		4 *lb.* 10 *oz.* 2 *pwt.* 12 *gr.*

(10.)

		Meters.		*Inches.*		*Feet.*
3.5 *m.*	=	3.5	,	or 137.795	,	or 11.48291⅔.
6½ *ft.*	=	1.9812	,	" 78.	,	" 6.5 .
482 *c.m.*	=	4.82	,	" 189.7634,		" 15.81361⅔.
286 *in.*	=	7.26441,		" 286.	,	" 23.83333⅓.
		17.56561 *m.*		691.5584 *in.*		57.62982/3 *ft.*

(11.) 5 *bbl.* 40 *qts.* = 167.5 *gal.*

6 *bush.* 100 *pt.* = 7.5625 *bush.* = $\frac{7.5625 \times 2150.4}{231}$ *gal.* = 70.4 *gal.* 167.5 *gal.* + 70.4 *gal.* = 237.9 *gal.*

Again, 167.5 *gal.* = $\frac{167.5 \times 231}{2150.4}$ *bush.* = 17.993 *bush.* 17.993 *bush.* + 7.5625 *bush.* = 25.5555 *bush.*

(12.) $\frac{7}{8}$ *cd.* = $\frac{7}{8}$ of 128 *cu ft.* = 112 *cu. ft.*

156 *cu. ft.* = 156 "

185643 *cu. in.* = 107.4322 "

$5\frac{3}{8}$ *stere* = $\frac{5\frac{3}{8} \times (39.37)^3}{1728}$ *cu. ft.* = 200.1152 "

575.5474 *cu. ft.*

Or, $\frac{7}{8}$ *cd.* = .875 *cd.*

156 *cu. ft.* = 1.21875 "

185643 *cu. in.* = $\frac{185643}{128 \times 1728}$ *cd.* = .839314 "

$5\frac{3}{8}$ *stere* = $\frac{5\frac{3}{8} \times (39.37)^3}{128 \times 1728}$ *cd.* = 1.5634 "

4.496464 *cd.*

(13.) \$300 = \$300

£56 = 56 × 4.8665 *dollars* = \$272.524

2000 *fr.* = .193 × 2000 " = \$386.

500 *M.* = .238 × 500 " = \$119.

\$1077.524

(14.) 1.25 *mi.* = 5280 × 1.25 *ft.* = 6600 *ft.*

7000 *rd.* = 7000 × 16.5 *ft.* = 115500 "

4385 *ft.* 4385 "

126485 *ft.*

(15.) $\frac{3}{4}$ *lb.* Apoth., or Troy, = $\frac{3}{4}$ × 5760 *gr.* = 4320 *gr.*

$5\frac{1}{2}$ *sc.* = $5\frac{1}{2}$ × 20 *gr.* = 110 "

$13\frac{3}{4}$ "

$4443\frac{3}{4}$ *gr.*

$\frac{3}{4}$ *lb.* Av. = $\frac{3}{4}$ × 7000 *gr.* = 5250 *gr.*

$5\frac{1}{2}$ *sc.* = $5\frac{1}{2}$ × 20 *gr.* = 110 "

$13\frac{3}{4}$ "

$5373\frac{3}{4}$ *gr.*

(16.)

$\frac{5}{8}$ *sq. yd.* $= \frac{5}{8} \times 9$ *sq. ft.* $= 5.625$ *sq. ft.*
$12\frac{4}{5}$ *sq. ft.* $= 12.6666+$ "
342.55 *sq. in.* $= 342.55 \div 144$ *sq. ft.* $= 2.3788+$ "
$20.6704+$ *sq. ft.*

(17.) This ordinary case of addition is inserted in order that the common principle of reducing to like denominations may be seen.

(18.)
7.25 *ft.* $= 87$ *in.*
3.82 *yd.* $= 137.52$ "
126.5 "
351.02 *in.*

(19.) The purpose of this is the same as of 17.

(20.)
$5\frac{1}{2}$ *mi.* $= 5.5$ *mi.*
488.3 *rd.* $= \frac{488.3}{320}$ *mi.* $= 1.5259$ "
$781\frac{1}{2}$ *ft.* $= \frac{781.5}{5280}$ *mi.* $= .0148$ "
7.0407 *mi.*

ADDITION IN THE LITERAL NOTATION.

PAGES 38–39.

(3.) $7x + 3x + 5x = (7 + 3 + 5)x = 15x.$

(4.) $5ab + 6ab + 10ab = (5 + 6 + 10)ab = 21ab.$

(6.)
$5x + 4m + 3ab$
$10x + 6m + 5ab$
$x + m + ab$
$16x + 11m + 9ab$

(7.)
$5ax + 6by + m$
$12ax + 4by + 7m$
$ax + 16by + 15m$
$18ax + 26by + 23m$

(8.)

$$\begin{array}{r} cd + 8ax + 10cy + 8ay \\ 4cd + ax + cy + 15ay \\ 3cd + 2ax + 5cy + 9ay \\ 7cd + 2ax + 9cy + 4ay \\ \hline 15cd + 13ax + 25cy + 36ay \end{array}$$

(9.)

$$\begin{array}{r} 3x^2 \\ 4x^2 \\ 7x^2 \\ x^2 \\ \hline 15x^2 \end{array}$$

(10.)

$$\begin{array}{r} 2x^2 + 3x + 1 \\ 5x^2 + 4x + 5 \\ x^2 + x + 4 \\ 3x^2 + x + 2 \\ \hline 11x^2 + 9x + 12 \end{array}$$

(12.)

$$\begin{array}{r} 5a^2 + 3a^3 + x \\ a^2 + 10a^3 + x \\ 4a^2 + 7a^3 + 5x \\ 31a^2 + 70a^3 + 19x \\ a^2 + x \\ a^3 + x \\ a^2 + a^3 \\ a^2 + a^3 + x \\ \hline 44a^2 + 93a^3 + 29x \end{array}$$

MULTIPLICATION.

PAGES 42–45.

(1–27.) There is nothing special in the solution of these examples. The main purpose is to show how the same principles apply throughout all cases in multiplication, and this line of thought is fully elaborated in the text. We will give here a single example in multiplying two numbers represented in the *Octary Scale*, by use of the Table on p. 11.

Ex.—Multiply 465_8 by 324_8.

OPERATION.

$$\begin{array}{r} 465 \\ 324 \\ \hline 2324 \\ 1152 \\ 1637 \\ \hline 177744 \end{array}$$

EXPLANATION. 4 times 5 are 24: write down the 4 and carry the 2. 4 times 6 are 30, and 2 are 32: write down the 2 and carry the 3. 4 times 4 are 20, and 3 are 23: write down the 3 and carry the 2 to the next place. Again, 2 times 5 are 12: write down the 2 under the figure by which you are multiply-

ing, and carry the 1. 2 times 6 are 14, and 1 are 15; write down the 5 and carry the 1. 2 times 4 are 10, and 1 makes 11; write down the whole. 3 times 5 are 17: write down the 7 and carry the 1, etc.

By the Addition Table on the same page, adding the 3d column, we have $7 + 5 = 14$, and 3 are 17.

MULTIPLICATION IN THE LITERAL NOTATION.

PAGES **46–48.**

(1.) $b^2 = 4^2 = 4 \times 4 = 16 : b^2 = 8^2 = 8 \times 8 = 64 : b^2 = (10)^2 = 10 \times 10 = 100 : b^2 = (\frac{1}{2})^2 = \frac{1}{2} \times \frac{1}{2} = \frac{1}{4} : (.03)^2 = .03 \times .03 = .009.$

(2.) $x^2 = 6^2 = 6 \times 6 = 36 : x^3 = 6^3 = 6 \times 6 \times 6 = 216.$

(3.) $2a^2 + 3x^2y - 4ay = 2 \times 7^2 + 3 \times 4^2 \times 3 - 4 \times 7 \times 3 = 2 \times 49 + 3 \times 16 \times 3 - 4 \times 7 \times 3 = 158.$

(4.) $a^2 + 2ab + b^2 = 4^2 + 2 \times 4 \times 20 + (20)^2 = 16 + 160 + 400 = 576 : a^2 + 2ab + b^2 = (30)^2 + 2 \times 30 \times 6 + (6)^2 = 900 + 360 + 36 = 1296.$

(5.) $x^2 - 2xy + y^2 = (60)^2 - 2 \times 60 \times 9 + (9)^2 = 3600 - 1080 + 81 = 2601.$

(6.) $a^3 + 3a^2b + 3ab^2 + b^3 = (20)^3 + 3 \times (20)^2 \times 5 + 3 \times 20 \times 5^2 + (5)^3 = 8000 + 6000 + 1500 + 125 = 15625 :$

$a^3 + 3a^2b + 3ab^2 + b^3 = (40)^3 + 3 \times (40)^2 \times 7 + 3 \times 40 \times 7^2 + 7^3 = 64000 + 33600 + 5880 + 343 = 103823.$

(7.) $a^3 - 3a^2b + 3ab^2 - b^3 = (600)^3 - 3 \times (600)^2 \times 40 + 3 \times 600 \times (40)^2 - (40)^3 = 216000000 - 43200000 + 2880000 - 64000 = 175616000 : a^3 - 3a^2b + 3ab^2 - b^3 = 6^3 - 3 \times 6^2 \times 2 + 3 \times 6 \times 2^2 - 2^3 = 216 - 216 + 72 - 8 = 64 : a^3 - 3a^2b + 3ab^2 - b^3 = 8^3 - 3 \times 8^2 \times 5 + 3 \times 8 \times 5^2 - 5^3 = 512 - 960 + 600 - 125 = 27.$

(9.)

$$\begin{array}{r} 5ax + 3b \\ 2a^2 \\ \hline 10a^2x + 6a^2b \end{array}$$

(10.)

$$\begin{array}{r} 3x^2 + xy + 2y^2 \\ 5x^2 \\ \hline 15x^4 + 5x^3y + 10x^2y^2 \end{array}$$

(11.)

$$\begin{array}{r} 3x^2y + 2x + 3 \\ 7xy \\ \hline 21x^3y^2 + 14x^2y + 21xy \end{array}$$

(12.)

$$\begin{array}{r} 12a^3 + 2ab^2 \\ 8a^2b \\ \hline 96a^5b + 16a^3b^3 \end{array}$$

(14.)

$$\begin{array}{r} a^2 + 2ab + b^2 \\ a + b \\ \hline a^3 + 2a^2b + ab^2 \\ a^2b + 2ab^2 + b^3 \\ \hline a^3 + 3a^2b + 3ab^2 + b^3 \end{array}$$

(16.)

$$\begin{array}{r} a^2x + 2by \\ 2a^2x + by \\ \hline 2a^4x^2 + 4a^2bxy \\ a^2bxy + 2b^2y^2 \\ \hline 2a^4x^2 + 5a^2bxy + 2b^2y^2 \end{array}$$

(17.)

$$\begin{array}{r} x + y \\ x + y \\ \hline xy + y^2 \\ x^2 + xy \\ \hline x^2 + 2xy + y^2 \end{array}$$

(18.)

$$\begin{array}{r} a + b \\ a + b \\ \hline ab + b^2 \\ a^2 + ab \\ \hline a^2 + 2ab + b^2 \\ a + b \\ \hline a^2b + 2ab^2 + b^3 \\ a^3 + 2a^2b + ab^2 \\ \hline a^3 + 3a^2b + 3ab^2 + b^3 \end{array}$$

NOTE.—Observe that in the Literal Notation it is quite immaterial whether we commence at the right or at the left to multiply. This is what we might expect from the nature of the notation, *as there is no established relation between the terms*, and hence nothing which corresponds to "carrying" in the Arabic Notation. What makes it more convenient to commence at the right than at the left in multiplying Arabic numbers?

(19.)

$$\begin{array}{r} x+1 \\ x+1 \\ \hline x^2+x \\ \quad x+1 \\ \hline x^2+2x+1 \end{array}$$

(20.)

$$\begin{array}{r} 3a^2+2by+c \\ \quad 3a^2y \\ \hline 9a^4y+6a^2by^2+3a^2cy \end{array}$$

(21.)

$$\begin{array}{r} x^2+x+1 \\ x^3+1 \\ \hline x^5+x^4+x^3+x^2+x+1 \end{array}$$

INVOLUTION.

PAGES 49–50.

(1–12.) These examples are simple applications of the processes of multiplication.

(13.) The 3d power of a number is a product containing the number 3 times as a factor; hence, taking this product 3 times as a factor, we have the original number involved 6 times as a factor, and hence have the 6th power. The 4th power of any number multiplied by the 5th power gives the 9th power, since the former contains the number 4 times as a factor, and the latter 9 times; hence the product of the 4th power and the 5th power of any number contains that number 9 times as a factor, and is therefore the 9th power.

(15.) $(23)^3 = 12167$. $(12167)^2 = 148035889$, which is therefore the 6th power of 23. Now $(148035889)^2 \times 12167 = (23)^{12} \times (23)^3 = (23)^{15} = 2666352354643912456 07$. There are thus 5 multiplications; 2 to obtain the cube, 1 more to obtain the 6th power, 1 more to obtain the 12th power, and then 1 to multiply the 12th power by the 3d power.

(*89.*) RULE.—*To involve a number to a high power,* produce such powers of the number that the sum of their indices make the index of the required power. The product of these powers is the required power.

To involve a monomial, involve the Arabic coefficient to the required power and to this power annex the letters of the monomial, having multiplied their exponents by the exponent of the required power.

PAGES 51–52.

(3.) $\sqrt{18664}$ contains 3 figures, since the square of the least number represented by 3 figures contains 5 figures, and the square of the least number represented by 4 figures contains 7 figures.

NOTE.—The above assumes that 18664 is a perfect square, as the problem states it to be. If, however, it were not, the same could be asserted of the integral part of the root, and for a like reason.

(4.) Answered like the above. So also those in EX. 1, (*91.*) The *reason* is the important thing.

(*92.*)

(3.) $(345)^2 = (340 + 5)^2 = (340)^2 + 2 \times 340 \times 5 + 5^2 = 115600 + 3400 + 25 = 119025$; $(4862)^2 = (4680 + 2)^2 = (4680)^2 + 2 \times 4680 \times 2 + 2^2 = 21902400 + 18720 + 4 = 21921124$.

PAGES 53–54.

(2.) $(86)^3 = (80 + 6)^3 = (80)^3 + 3 \times (80)^2 \times 6 + 3 \times 80 \times 6^2 + 6^3 = 512000 + 115200 + 8640 + 216 = 636056$: The others are similar.

(3.) $(283)^3 = (280 + 3)^3 = (280)^3 + 3 \times (280)^2 \times 3 + 3 \times (280) \times 3^2 + 3^3 = 21952000 + 705600 + 7560 + 27 = 22665187$: The others in this and 4 and 5 are similar.

The examples on page 54 require simply an application of the demonstration at the top of that page, which need not be repeated here. *It will afford a good written exercise to require that the pupils, with books closed, write out such a demonstration.*

SUBTRACTION.

PAGES **57–59.**

(***99.***) 1. Only abstract numbers, or concrete numbers representing things of the same kind, can be subtracted the one from the other.

2. Only like orders or denominations can be directly taken the one from the other.

3. In subtracting related orders or denominations it is practically most convenient to proceed from the lowest order or denomination to the highest; since, by so doing, we are enabled, when the number represented in any given order or denomination in the subtrahend exceeds the corresponding number in the minuend, to take a unit of the next higher order of the minuend and adding it to the number of this order in the minuend, effect the subtraction.

(1–6.) The *operation* of solving these is presumed to be familiar. *The purpose is to show the identity of principles involved.* It will afford an excellent exercise for the pupil to write out in full demonstrations of the processes, as in Ex's 1 and 2, and compare them. There should be no difference except in the use of the words *order* and *denomination.*

(7.)

$$\begin{aligned} 247.3\ ft. &= 247.3 \times 12\ in. &&= 2967.6\ in. \\ 5\tfrac{1}{2}\ m. &= 5\tfrac{1}{2} \times 39.37\ in. &&= \underline{216.535\ in.} \\ & && 2751.065\ in. \end{aligned}$$

(8.)

$$\begin{aligned} 7\ gal. &= 6\ gal.\ 4\ qt., \quad \text{or } 4\ gal.\ 12\ qt. \\ & \qquad \underline{2\ qt.} \qquad\qquad \underline{10\ qt.} \\ & 6\ gal.\ 2\ qt. \qquad 4\ gal.\ 2\ qt. \end{aligned}$$

(9–16.) These *operations* are presumed to be familiar; but attention should be directed to the identity of princi-

ples involved. Thus in solving the 12th, let it be shown that in taking $\frac{8}{11}$ from $7\frac{8}{8}$, exactly the same principles are involved as in subtracting 8 from 73. We give the illustration in parallel columns.

73	As we cannot take 8
8	units from 3 units we
65	take 1 ten from the 7 tens, and adding it to the 3 have 13. From this taking 8 units there remain 5 units. The 7 having been diminished by 1, 6 remains.

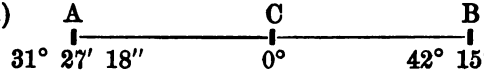
$7\frac{8}{8}$	As we cannot take $\frac{8}{11}$
$\frac{8}{11}$	from $\frac{8}{8}$, we take 1 unit
$6\frac{88}{88}$	from the 7, and adding it to the $\frac{8}{8}$ have $\frac{8}{8}$, or $\frac{88}{88}$. From this taking $\frac{8}{11}$, or $\frac{40}{88}$, there remains $\frac{48}{88}$. The 7 having been diminished by 1, 6 remains.

(*102.*)

(1.) As the longitudes are both east, the difference is found by subtracting the numbers.

$$\begin{array}{r} 42^\circ\ 15'\ 20'' \\ 31^\circ\ 27'\ 18'' \\ \hline 10^\circ\ 48'\ 2'' \end{array}$$

(2.)



A being 31° 27′ 18″ west of C (the 0 of longitude), and B 42° 15′ 20″ east, the difference in longitude is the *sum* of the two numbers.

$$\begin{array}{r} 42^\circ\ 15'\ 20'' \\ 31^\circ\ 27'\ 18'' \\ \hline 73^\circ\ 42'\ 38'' \end{array}$$

(3.) The sum of the two numbers 26° 18′ 3″ and 35°, or 61° 18′ 3″.

(4.) The sum of 23° and 35°, or 58°.

SUBTRACTION IN THE LITERAL NOTATION.

PAGES **59–60.**

(3.)

$$\begin{array}{r} 17a^2x \\ 13a^2x \\ \hline 4a^2x \end{array} \qquad \begin{array}{r} 9by^2 \\ 4by^2 \\ \hline 5by^2 \end{array}$$

EXPLANATION. 17 times any quantity, as a^2x, minus 13 times *the same* quantity, leaves 4 times that quantity. Hence, $17a^2x - 13a^2x = 4a^2x$.

In like manner, 9 times any quantity, as by^2, minus 4 times the same quantity, leaves 5 times that quantity. Hence, $9by^2 - 4by^2 = 5by^2$.

(5.)

$$\begin{array}{r} 3c^2x + 4ay + 10b^2 \\ c^2x + 4ay + 3b^2 \\ \hline 2c^2x \qquad\quad + 7b^2 \end{array}$$

Explanation the same as given to Ex. 4 in the text-book.

Observe that we effect the subtraction in the same manner as in the Arabic notation, by taking the several parts which make up the subtrahend out of the minuend, taking each part of the subtrahend from a similar part of the minuend when we can. Similar terms in the literal notation bear some analogy to like orders in the Arabic.

(7.) $3a - 4b$. We can do nothing more than indicate this operation, as we know nothing of the relative values of $3a$ and $4b$.

(9.) These operations can only be indicated thus, $x^3 - x^2$, $x^2 - x$, $a - 1$, $1 - x$, and for reasons like that given above. In the case of the first two, we do indeed know something of the relation between x^3 and x^2, and x^2 and x; but nothing which indicates what the *difference* is.

ADDITION OF POSITIVE AND NEGATIVE QUANTITIES.

PAGES **61–63.**

(2.) $\$3ax + \$2ax = \$5ax$.

(3.)

$$\begin{array}{r} 4by - 2ax \\ 5by - 4ax \\ \hline 9by - 6ax \end{array}$$

(4.)

$$\begin{array}{r} 5x^2 + 2y^3 - 4cy \\ 3x^2 + 5y^3 - 2cy \\ \hline 8x^2 + 7y^3 - 6cy \end{array}$$

(7.) The *difference* between $7ax$ degrees below 0°, and $4ax$ degrees above is the *sum* of $7ax$ and $4ax$, or $11ax$, as explained under subtraction of Arabic numbers (***102***).

(8.)

$$\begin{array}{r} 2a - 8x^2 \\ -3a + x^2 \\ -4a - 2x^2 \\ -a + 4x^2 \\ \hline -6a - 5x^2 \end{array}$$

In adding we consider that $+4x^2$ and $-2x^2$ is $+2x^2$; $+2x^2$ and $+x^2$ is $+3x^2$; and $+3x^2$ and $-8x^2$ is $-5x^2$. Hence the sum of the terms in the first column is $-5x^2$. In like manner $-a$, $-4a$, and $-3a$ make $-8a$. Then the sum of $-8a$ and $+2a$ is $-6a$.

(9.)

$$\begin{array}{r} 2 - x + 4y^3 \\ -30 - x - 2y^3 \\ 3 + 3x - y^3 \\ 1 - 2x + 3y^3 - 10z \\ \hline -24 - x + 4y^3 - 10z \end{array}$$

(10.)

$$\begin{array}{r} 3x + 5y - 6az \\ -2x - 8y - 9az \\ 20x + 2y - 3az \\ x - y + az - 4 \\ \hline 22x - 2y - 17az - 4 \end{array}$$

(11.)

$$\begin{array}{r} 3 - 2y + z \\ 5 + 4y - 2z \\ 2 - y - z \\ -10 - y + 2z \\ \hline 0 \quad 0 \quad 0 \end{array}$$

(12.)

$$\begin{array}{r} 3ax - 6 \\ 4ax + 2 \\ 3ax + 6 \\ -2ax + 7 \\ ax + 1 \\ \hline 9ax + 10 \end{array}$$

(13.)

$$\begin{array}{r} 4x - a \\ -x + 3a \\ 4x + a \\ -3x + 7a \\ \hline 4x + 10a \end{array}$$

(14.)

$$\begin{array}{r} 2 + 2z \\ 3z \\ 8 + 3z \\ 5 - 2z \\ 4z \\ \hline 15 + 10z \end{array}$$

(15.)

$$\begin{array}{r} 4x \\ -3x + 4 \\ x \\ -2x - 5 \\ 3x + 1 \\ -5x \\ \hline -2x \end{array}$$

(16.)

$$\begin{array}{rr} 2x + y + 9 & \\ -x - y - 9 & \\ 3x & +10xy \\ \hline 4x & +10xy \end{array}$$

(17.)	(18.)	(19.)
$5a + 3by - 4c$	$3a + x + 10$	$5a^2 - 3b$
$2a - 5by + 6c$	$-5a + 2x - 15$	$4a^2 - 7b$
$a - 4by - 2c$	$-4a - 10x + 21$	$7a^2 + 3b$
$8a - 6by$	$-6a - 7x + 16$	$-5a^2 - 9b$
		$11a^2 - 16b$

SUBTRACTION OF POSITIVE AND NEGATIVE QUANTITIES.

PAGES **63, 64.**

(1.) The same quantities with their signs changed.

$$\begin{array}{r} 4ax - by + x^2 \\ -2ax + by - x^2 \\ \hline 0 \quad 0 \quad 0 \end{array}$$

(2.) Same as (1).

(8.) To subtract $-2ax + 5b^2y - c$ from $7ax + 2b^2y - 3c$, is to destroy a portion of the latter equal to the former. This is effected by adding the subtrahend with its signs changed.

$$\begin{array}{r} 7ax + 2b^2y - 3c \\ 2ax - 5b^2y + c \\ \hline 9ax - 3b^2y - 2c \end{array}$$

NOTE.—It is not customary to change the signs in practice, but only to conceive them changed.

(9.)	(10.)	(11.)
$8ab - 6a^2z^2 - a^3y$	$3a - 2b$	$4 - 3xy$
$11a^2z^2 + 3a^3y$	$-2a + b$	$-1 + xy$
$8ab - 17a^2z^2 - 4a^3y$	$5a - 3b$	$5 - 4xy$

MULTIPLICATION OF POSITIVE AND NEGATIVE QUANTITIES.

PAGES **65, 66.**

There is nothing new in this section save the law of the signs in multiplication of positive and negative quantities; this, therefore, should be carefully attended to. Article (***108***) with the demonstration, is a full exposition of this law, but it will need to be dwelt upon and amplified for the ordinary student.

(1.)

$$\begin{array}{r} 5ac - 3a^2 + by \\ + 4a^2y \\ \hline 20a^3cy - 12a^4y + 4a^2by^2 \end{array}$$

The signs of $4a^2y$ and by being *alike*, the product is $+$. The signs of $4a^2y$ and $3a^2$ being *unlike*, the product is $-$ (***108***).

$$\begin{array}{r} 5ac - 3a^2 + by \\ - 4a^2y \\ \hline -20a^3cy + 12a^2y - 4a^2by^2 \end{array}$$

The signs of $4a^2y$ and by being *unlike*, the product is $-$. The signs of $4a^2y$ and $3a^2$ being *alike*, the product is $+$ (***108.***)

(2.)

$$\begin{array}{r} 3a - 2b \\ 3a + 2b \\ \hline 6ab - 4b^2 \\ 9a^2 - 6ab \\ \hline 9a^2 \quad - 4b^2 \end{array}$$

(3.)

$$\begin{array}{r} 3a - 2b \\ 3a - 2b \\ \hline - 6ab + 4b^2 \\ 9a^2 - 6ab \\ \hline 9a^2 - 12ab + 4b^2 \end{array}$$

(4.)

$$\begin{array}{r} x^2 - 2xy + y^2 \\ x - y \\ \hline x^3 - 2x^2y + xy^2 \ * \\ - x^2y + 2xy^2 - y^3 \\ \hline x^3 - 3x^2y + 3xy^2 - y^3 \end{array}$$

The 5th may also be solved by multiplying the square of $a - b$ (*i. e.* $a^2 - 2ab + b^2$) by itself. See Involution.

(5.)

$$\begin{array}{r} a - b \\ a - b \\ \hline a^2 - ab \ * \\ - ab + b^2 \\ \hline a^2 - 2ab + b^2 \\ a - b \\ \hline a^3 - 2a^2b + ab^2 \\ - a^2b + 2ab^2 - b^3 \\ \hline a^3 - 3a^2b + 3ab^2 - b^3 \\ a - b \\ \hline a^4 - 3a^3b + 3a^2b^2 - ab^3 \\ - a^3b + 3a^2b^2 - 3ab^3 + b^4 \\ \hline a^4 - 4a^3b + 6a^2b^2 - 4ab^3 + b^4 \end{array}$$

* In the literal notation it is perhaps a little more convenient to write the partial products if we begin the multiplication at the left.

(6.)

$$\begin{array}{r} 1-r \\ 1-r \\ \hline 1-r \\ -r+r^2 \\ \hline 1-2r+r^2 \end{array} \qquad \begin{array}{r} 1-r \\ 1+r \\ \hline 1-r \\ +r-r^2 \\ \hline 1-r^2 \end{array}$$

(7.)

$$\begin{array}{r} 1+r \\ 1+r \\ \hline 1+r \\ r+r^2 \\ \hline 1+2r+r^2 \\ 1+2r+r^2 \\ \hline 1+2r+r^2 \\ 2r+4r^2+2r^3 \\ r^2+2r^3+r^4 \\ \hline 1+4r+6r^2+4r^3+r^4 \\ 1+2r+r^2 \\ \hline 1+4r+6r^2+4r^3+r^4 \\ 2r+8r^2+12r^3+8r^4+2r^5 \\ r^2+4r^3+6r^4+4r^5+r^6 \\ \hline 1+6r+15r^2+20r^3+15r^4+6r^5+r^6 \end{array}$$

In involving $1+r$ to the 6th power we have multiplied the square by the square, thus producing the 4th power; and the 4th power by the square, thus producing the 6th power. See Involution.

$(1-r)^2 = 1-2r+r^2$; $(1-2r+r^2)^2 = (1-r)^4 = 1-4r+6r^2-4r^3+r^4$: $(1-4r+6r^2-4r^3+r^4) \times (1-r) = (1-r)^4\,(1-r) = (1-r)^5 = 1-5r+10r^2-10r^3+5r^4-r^5$.

(8.)

$$\begin{array}{r} 1+rp \\ \frac{r}{p} \\ \hline \frac{r}{p}+r^2 \end{array}$$

To multiply rp by $\frac{r}{p}$ we proceed as in the Arabic arithmetic, *i. e.* multiply rp by the numerator r, giving r^2p, and then divide this product by p, giving r^2.

(9.)

$$\begin{array}{r} ar \ -a \\ 1 \ -\frac{a}{r} \\ \hline -a^2+\frac{a^2}{r}+ar-a \end{array}$$

(10.)

$$\begin{array}{r} \frac{b^2}{a^2}-x^2y^2 \\ \frac{a^2}{x^2} \\ \hline \frac{b^2}{x^2}-a^2y^2 \end{array} = \frac{b^2-a^2y^2x^2}{x^2}$$

(11.) $$\frac{a}{b} \times \frac{x}{y} = \frac{ax}{by}.$$

(12.)

$$\begin{array}{r} 1-\frac{a}{b} \\ 1+\frac{x}{y} \\ \hline 1-\frac{a}{b}+\frac{x}{y}-\frac{ax}{by} \end{array}$$

$$1-\frac{a}{b}=\frac{b-a}{b}.$$

$$1+\frac{x}{y}=\frac{y+x}{y}.$$

$$\frac{b-a}{b} \times \frac{y+x}{y} = \frac{(b-a)\,(y+x)}{by} = \frac{by-ay+bx-ax}{by}$$

(13.) The process of reducing $1-\frac{a}{b}+\frac{x}{y}-\frac{ax}{by}$ to the improper fraction $\frac{by-ay+bx-ax}{by}$ shows them to be equivalent. This may also be shown by reducing $\frac{by-ay+bx-ax}{by}$ to a mixed number. Thus by is contained in by 1 time, in $-ay$ $-\frac{a}{b}$ times, in $+bx$ $\frac{x}{y}$

times, and in $-ax$ $-\frac{ax}{by}$ times; hence $\frac{by-ay+bx-ax}{by}$ $=1-\frac{a}{b}+\frac{x}{y}-\frac{ax}{by}$.

THE PARENTHESIS.

Pages 66, 67.

(1.) $a(a-b)=a^2-ab$; $(a-b)(a+b)=a^2-ab+ab-b^2=a^2-b^2$; $(a+c)^2=(a+c)(a+c)=a^2+ac+ac+c^2=a^2+2ac+c^2$.

(2.)

$$\begin{array}{r} a^2+2ax+x^2 \\ a^2-2ax+x^2 \\ \hline 4ax \end{array}$$

(3.) $5ax-2(c+b)=5\times 6\times 2-2(3+5)=60-2\times 8=60-16=44$. Or, $5ax-2(c+b)=5ax-(2c+2b)=5ax-2c-2b=60-6-10=60-16=44$.

(4.) $3x^2-(a^2-b^2)+2c=3\times 4-(36-25)+2\times 3=12-11+6=7$. Or, $3x^2-(a^2-b^2)+2c=3x^2-a^2+b^2+2c=3\times 4-36+25+2\times 3=12-36+25+6=43-36=7$.

(5.) $14-(5-2)-4(3+5)+6(7-3)=14-3-32+24=3$. Or, $14-(5-2)-4(3+5)+6(7-3)=14-(5-2)-(12+20)+(42-18)=14-5+2-12-20+42-18=58-55=3$.

(6.) $(9-3)(7+2)=6\times 9=54$. Or, $(9-3)(7+2)=63-21+18-6=81-27=54$: $3(4-2)-5(7-1)\times\frac{2}{5}=3\times 2-5\times 6\times\frac{2}{5}=6-20=-14$. Or, $3(4-2)-5(7-1)\times\frac{2}{5}=(12-6)-(\frac{70}{5}-\frac{10}{5})=12-6-\frac{70}{5}+\frac{10}{5}=\frac{46}{5}-\frac{82}{5}=-\frac{42}{5}=-14$.

(7.) $\frac{1}{2} + \frac{1}{2} + \frac{1}{2} \times \frac{2}{3} \div \frac{1}{2} = \frac{1}{2} + \frac{1}{2} + \frac{2}{3} = 1\frac{2}{3}$. $(\frac{1}{2} + \frac{1}{2} + \frac{1}{2}) \times \frac{2}{3} \div \frac{1}{2} = \frac{3}{2} \times \frac{1}{3} = 2$.

(8.) $5 - \frac{4+3}{11} = \frac{55-4-3}{11} = \frac{48}{11} = 4\frac{4}{11}$. Or, $5 - \frac{4+3}{11}$ $5 - \frac{7}{11} = \frac{55}{11} - \frac{7}{11} = \frac{48}{11} = 4\frac{4}{11}$: $5 - \frac{4+3}{11} = \frac{55}{11} - \frac{4+3}{11}$ $= \frac{55-(4+3)}{11} = \frac{55-4-3}{11}$, since after having reduced to forms having a common denominator $4 + 3$ is *to be subtracted* from 55; and this is done by adding -4 and -3, *i. e.* by changing the signs of the subtrahend and adding it to the minuend. See (***107.***)

(10.) $3ax - (c - 15d) = 3ax - c + 15d$: $4y + (3 - ab) = 4y + 3 - ab$: $(6ay - 3cd) - 5 = 6ay - 3cd - 5$: $7ax + 2by - (10d + 4) = 7ax + 2by - 10d - 4$. The substitutions are readily effected as in preceding cases.

DIVISION.

Pages 71, 72.

(1–6.) These are ordinary operations in division, given for the purpose of illustrating the Principles given in (***113***).

(7.) 3 *yd.* 5 *ft.* 2 *in.* = 170 *in.* $\frac{2}{3}$ *m.* = $\frac{2}{3}$ of 39.37 *in.* = 26.246 *in.* Hence (3 *yd.* 5 *ft.* 2 *in.*) $\div$ $\frac{2}{3}$ *m.* = 170 *in.* $\div$ 26.246 *in.* = 6.477 +. $4\frac{3}{8} = \frac{35}{8}$, and .25 = $\frac{2}{8}$. Hence $4\frac{3}{8} \div .25 = \frac{35}{8} \div \frac{2}{8}$, or $35 \div 2 = 17\frac{1}{2}$. In the former case dividend and divisor were reduced to the same denomination, viz., *inches;* in the latter, dividend and divisor were reduced to the same denomination, viz., *eighths.* In this view of a fraction the numerator is conceived as the number proper, and the denominator is conceived as telling the denomination of the number represented by the numerator.

(8.) 7 *lb.* 12 *oz.* Av. = 54250 *gr.*, and 1 *lb.* 20 *pwt.* = 6240 *gr.* Hence (7 *lb.* 12 *oz.* Av.) $\div$ (1 *lb.* 20 *pwt.*) = 54250 $\div$ 6240 = 8.69+.

(9.) To divide $\frac{4}{7}$ of a ton by 3 is to take $\frac{1}{3}$ of it, or to separate it into 3 equal parts and take one of those parts; but what is meant by dividing $\frac{4}{7}$ of a ton by $\frac{2}{3}$? We may not say that it is finding how many times $\frac{2}{3}$ is contained in $\frac{4}{7}$ of a ton, for $\frac{2}{3}$ is an abstract number, while $\frac{4}{7}$ is concrete. The thing signified is that we are to take some certain part of $\frac{4}{7}$ of a ton, *i. e.*, $\frac{1}{5}$ of 5 times $\frac{4}{7}$ of a ton, which gives $\frac{4}{7}$ of a ton.

(11.) $\frac{1}{5}$ is contained in 1 5 times, as by definition $\frac{1}{5}$ is one of the 5 equal parts of unity. Then as $\frac{2}{5}$ is 3 times $\frac{1}{5}$, it is contained in 1 $\frac{2}{5}$ as many times as is $\frac{1}{5}$, or $\frac{1}{3}$ of 5 times $= \frac{5}{3}$ times. Or, by PRIN. 4, we may consider the 1 as 5 *fifths*, reducing it to the same denomination as the divisor, and then we have 5 fifths $\div$ 3 fifths $= 5 \div 3 = \frac{5}{3}$, just as \$5 $\div$ \$3 $= 5 \div 3 = \frac{5}{3}$.

(12.) $\frac{3}{8}$ *cd.* $= 48$ *cu. ft.* 48 *cu. ft.* $\div$ $5\frac{1}{3}$ *cu. ft.* $= 9$.

(12.) The *process* is presumed to be familiar; let careful attention be given to the *rationale*. In this case we may consider 12.5 as 125 *tenths*, and .5 as 5 *tenths*. Hence we have 12.5 $\div$ .5 $=$ 125 *tenths* $\div$ 5 *tenths* $=$ 125 $\div$ 5 $=$ 25; just as 125 *pounds* $\div$ 5 *pounds* $=$ 125 $\div$ 5 $=$ 25. Again, 12.5 $\div$ .05 $=$ 1250 *hundredths* $\div$ 5 *hundredths* $=$ 1250 $\div$ 5 $=$ 250. So also 12.5 $\div$ 5 $=$ 125 *tenths* $\div$ 50 *tenths* $=$ 125 $\div$ 50 $=$ 2.5. The *rationale* consisting in the reduction of both dividend and divisor to like denominations.

(***114.***) It is probable that this rule for the division of decimals will be found most simple in theory, and easy in practice for *beginners*.

(14.) .78 $\div$ 3 $=$ 78 $\div$ 300 $=$ .26 : .78 $\div$ 30 $=$ 78 $\div$ 3000 $=$ .026 : .78 $\div$ .3 $=$ 78 $\div$ 30 $=$ 2.6 : .78 $\div$ .003 $=$ 780 $\div$ 3 $=$ 260.

(15.) 17.63 $\div$ 2.4 $=$ 1763 $\div$ 240 $=$ 7.34 : 17.63 $\div$ .12 $=$ 1763 $\div$ 12 $=$ 146.91 : 17.63 $\div$ 52 $=$ 1763 $\div$

5200 = .339+ : 17.63 ÷ .251 = 17630 ÷ 251 = 70.24 : 17.63 ÷ 1.0065 = 176300 ÷ 10065 = 17.516+.

(*116.*) See (*86*) and (*91*) in the ELEMENTS OF ARITHMETIC.

(*120.*) 1. The numerator being the dividend and the denominator the divisor, while the value of the fraction is the quotient, dividing numerator and denominator by the same number is dividing dividend and divisor, and hence does not alter the value of the fraction (PROP. 1).

2. Reducing an improper fraction to a whole or mixed number is but performing the indicated operation.

3. To reduce a whole number to a fraction having a given denominator, we multiply the whole number by the proposed denominator, and then writing the denominator underneath the product indicate division by it; hence the value of the original number has not been changed since it has been multiplied and divided by the same number. In reducing a mixed number to an improper fraction we remember that such an expression as $4\frac{2}{3}$ means $4 + \frac{2}{3}$; then reducing the 4 to thirds as just explained, we perform the addition; which we can now do, as the parts have a common divisor. See next paragraph.

5. In adding or subtracting fractions, the reduction to forms having a common denominator may be explained as putting the unexecuted problems in division (*i. e.*, the several fractions) into forms having a common divisor, and then, by PROP. 4, since the sum of the quotients is the same as the quotient of the sum, we can add the several dividends, and divide the sum by the common divisor, or indicate the divisor.

7. In this view of a fraction the explanation of *Involution* is the same as that given in the text-book for multiplication. Thus $(\frac{2}{3})^2 = \frac{2}{3} \times \frac{2}{3}$. Now $\frac{2}{3} \times 2 = \frac{4}{3}$, since doubling the dividend doubles the quotient, or value of the fraction. But our divisor was only $\frac{1}{3}$ of 2; hence this pro-

duct is 3 times as large as it should be. We therefore divide $\frac{4}{5}$ by 3, which is done by multiplying the divisor by 3, producing $\frac{4}{15}$.

Evolution being the converse of Involution, if a fraction is involved by involving numerator and denominator separately, the root of a fraction will be extracted by extracting the root of each term separately.

NOTE.—It will be found that many attempts to explain the operations in fractions upon the conception of a fraction as an unexecuted problem in division are defective, in that more or less of the processes are made to depend on the other conception, viz., that the fraction represents a certain number of equal parts of some quantity taken as a unit. The same conception should be retained throughout. No doubt the latter is the best for elementary purposes; but the former is required in order to give the breadth of conception essential to the genius of the literal notation. See ALGEBRA.

DIVISION IN THE LITERAL NOTATION.

PAGES **76–79.**

(7.) $12a^2b \div 4ab^2 = \frac{12a^2b}{4ab^2} = \frac{4ab \times 3a}{4ab \times b} = \frac{3a}{b}$.

The signs being alike the quotient is $+$.

(8.) $-16a^2x \div 4ax^2 = \frac{-16a^2x}{4ax^2} = -\frac{4ax \times 4a}{4ax \times x} = -\frac{4a}{x}$.

The signs being unlike the quotient is $-$.

(9.) $144a^3b^2x \div -24a^2b^4 = \frac{144a^3b^2x}{-24a^2b^4} = -\frac{24a^2b^2 \times 6ax}{24a^2b^2 \times b^2} = -\frac{6ax}{b^2}$.

(10.) $\frac{a^3b^2c^3}{a^3b^2cx^2} = \frac{a^3b^2c \times c^2}{a^3b^2c \times x^2} = \frac{c^2}{x^2}$.

(11.) $a \div b = \frac{a}{b}$. There being no common factor in the terms, the operation can only be indicated.

(12.) $\frac{-16a^2}{4a^2x} = -\frac{4a^2 \times 4}{4a^2 \times x} = -\frac{4}{x}$.

(13.) $\frac{-6mx^2}{-24nx^4} = \frac{6x^2 \times m}{6x^2 \times 4nx^2} = \frac{m}{4nx^2}$. The signs being both — (both alike), the quotient is +.

(14.) $\frac{256a^5b^4c^3}{16a^5b^2c^3mn} = \frac{16a^5b^2c^3 \times 16b^2}{16a^5b^2c^3 \times mn} = \frac{16b^2}{mn}$.

(15.) $\frac{15ax^2y^2}{45a^2bx^3y^2} = \frac{15ax^2y^2 \times 1}{15ax^2y^2 \times 3abx} = \frac{1}{3abx}$.

(16.) $\frac{39496a^3x^3}{69118a^4x^2y} = \frac{9874a^3x^2 \times 4x}{9874a^3x^2 \times 7ay} = \frac{4x}{7ay}$.

(17.) $\frac{58760x}{66105x^3y} = \frac{7345x \times 8}{7345x \times 9x^2y} = \frac{8}{9x^2y}$.

(18.) $\frac{25194a^2x}{88179a^2x^2} = \frac{12597a^2x \times 2a}{12597a^2x \times 7x} = \frac{2a}{7x}$.

(21.)

$$-3ax \,)\, 12a^2x - 18a^3x^2 \;(\; -4a + 6a^2x$$

By the same principle as in Arabic division, we effect the division by finding how many times the divisor is contained in the parts of the dividend, and adding the quotients. $-3ax$ is contained in $12a^2x$ $-4a$ times, the — sign being a consequence of the unlike signs in divisor and dividend. Again, $-3ax$ is contained in $-18a^3x^2$ $6a^2x$ times, the sign of the quotient being +, since divisor and dividend have like signs. Finally, adding these partial quotients by connecting them with their own signs, we have for the entire quotient $-4a+6a^2x$.

The solution of the following examples should be explained by the pupil in a similar manner.

(22.)

$$4amx \,)\, 16am^2x - 8a^2mx^2 + 4a^3m^3x^3 \;(\; 4m - 2ax + a^2m^2x^2$$

(*123.*) RULE.—*To divide a polynomial by a monomial,* divide each term of the polynomial by the divisor, and connect the partial quotients by their own signs.

(23.)
$$5x \,)\, 10xz + 15xy \quad \underline{\qquad}$$
$$2z + 3y$$

(24.)
$$3x \,)\, 15ax - 27x \quad \underline{\qquad}$$
$$5a - 9$$

(25.)
$$9x \,)\, 18x^2 - 9x \quad \underline{\qquad}$$
$$2x - 1$$

(26.)
$$-bc \,)\, abc - bcd - bcx \quad \underline{\qquad}$$
$$-a + d + x$$

(27.)
$$3x \,)\, 3x + 6x^2 + 3ax - 15x \quad \underline{\qquad}$$
$$1 + 2x + a - 5$$

(28.)
$$3ab \,)\, 3abc + 12abx - 9a^2b \quad \underline{\qquad}$$
$$c + 4x - 3a$$

(29.)
$$ab \,)\, 40a^3b^2 + 60a^2b^2 - 17ab \quad \underline{\qquad}$$
$$40a^2b + 60ab - 17$$

(30.)
$$-5ac \,)\, 15a^2bc - 10acx^2 + 5ad^2c \quad \underline{\qquad}$$
$$-3ab + 2x^2 - d^2$$

(31.)
$$5a \,)\, 20ax + 15ax^2 + 10ax - 5a \quad \underline{\qquad}$$
$$4x + 3x^2 + 2x - 1$$

(32.)
$$3xy \,)\, 12xy - 9x^2y^2 - 3ax \quad \underline{\qquad}$$
$$4 - 3xy - \frac{a}{y}.$$

In the last $-3ax$ divided by $3xy$ is $-\frac{3ax}{3xy}$, whence the factors 3 and x cancel, leaving $-\frac{a}{y}$ as the quotient.

(33.)

$$7a^4x^3 \,)\, 5a^2x - 4a^3x^3$$

$$\frac{5}{7a^2x^2} - \frac{4}{7a}$$

(34.)

$$\frac{x}{y} \div \frac{a}{b} = \frac{x}{y} \times \frac{b}{a} = \frac{bx}{ay}.$$

In the 33*d*, $5a^2x$ divided by $7a^4x^3$ is $\frac{5a^2x}{7a^4x^3}$, whence cancelling a^2x, we have $\frac{5}{7a^2x^2}$. In like manner we get $-\frac{4a^3x^3}{7a^4x^3}$ $= -\frac{a^3x^3 \times 4}{a^3x^3 \times 7a} = -\frac{4}{7a}$.

In the 34th the ordinary process of dividing one fraction by another is applied, viz., invert the divisor and multiply.

(35.) $\frac{12a^2x^2}{35b^3y^3} \div \frac{6ax^2}{7b^2y^3} = \frac{2a}{5b}$. In this case the division is best effected by dividing the numerator of the dividend by the numerator of the divisor, and the denominator of the dividend by the denominator of the divisor.

(37.) $5a + \frac{3b}{2x} = \frac{10ax + 3b}{2x}$. $5a = \frac{10ax}{2ax}$.

(38.) $5a - \frac{3b}{2x} = \frac{10ax - 3b}{2x}$.

(39.) $1 - \frac{x}{y} = \frac{y - x}{y}$, and $1 + \frac{x}{y} = \frac{y + x}{y}$; $\therefore \left(1 - \frac{x}{y}\right) \div \left(1 + \frac{x}{y}\right) = \frac{y - x}{y} \div \frac{y + x}{y} = \frac{y - x}{y + x}$. Divide the denominator of the dividend by the denominator of the divisor, and since the numerator of the dividend is not divisible by the numerator of the divisor, indicate the operation by writing the latter under the former. (The last process may also be conceived as multiplying the denominator of the dividend by the numerator of the divisor, which is equivalent to dividing the numerator.)

(40.) $\frac{2a}{3b} - \frac{x}{y} = \frac{2ay - 3bx}{3by}$, and $\frac{a}{b} + \frac{2x}{3y} = \frac{3ay + 2bx}{3by}$; $\therefore \left(\frac{2a}{3b} - \frac{x}{y}\right) \div \left(\frac{a}{b} + \frac{2x}{3y}\right) = \frac{2ay - 3bx}{3by} \div \frac{3ay + 2bx}{3by} = \frac{2ay - 3bx}{3ay + 2bx}$. The operation of division is explained as in the 39th.

EVOLUTION.—SQUARE ROOT.

PAGE 80.

(1.) $16 = (2 \times 2) \times (2 \times 2)$; $\therefore \sqrt{16} = 2 \times 2 = 4$; $32 = 2 \times 2 \times 2 \times 2 \times 2$; $\therefore \sqrt[5]{32} = 2$; $1728 = 12 \times 12 \times 12$; $\therefore \sqrt[3]{1728}$ 12.

The leading purpose of this article is to illustrate and fix the idea that *a root is one of the equal factors into which a number is conceived to be resolved.*

(3.)

3) 11025	10 × 10) 32400
3) 3675	2) 324
5) 1225	2) 162
5) 245	9) 81
5) 49	9
7	
$\therefore 11025 = (3 \times 5 \times 7) \times (3 \times 5 \times 7)$; and $\sqrt{11025} = 3 \times 5 \times 7 = 105$.	$\therefore 32400 = (10 \times 2 \times 9) (10 \times 2 \times 9)$; and $\sqrt{32400} = 10 \times 2 \times 9 = 180$.

$\sqrt{245025} = \sqrt{(5 \cdot 3 \cdot 3 \cdot 11)\ (5 \cdot 3 \cdot 3 \cdot 11)} = 5 \cdot 3 \cdot 3 \cdot 11 = 495$. $\sqrt{48841} = \sqrt{(17 \cdot 13)\ (17 \cdot 13)} = 17 \cdot 13 = 221$.

$10 \times 10 \,)\, 145600$
$2 \,)\, 1456$
$2 \,)\, 728$
$2 \,)\, 364$
$2 \,)\, 182$
$7 \,)\, 91$
13

Since 7 and 13 are each contained in this number but *once* as a factor, the number is not a perfect square. To such cases this method is not applicable, since it affords no method of approximation.

(5.) $\sqrt[3]{46656} = \sqrt{(2\cdot2\cdot3\cdot3)\ (2\cdot2\cdot3\cdot3)\ (2\cdot2\cdot3\cdot3)} = 2\cdot2\cdot3\cdot3 = 36$. $\sqrt[3]{621875} = \sqrt[3]{5\cdot5\cdot5\cdot5\cdot5\cdot199}$. As 621875 is not revolvable into 3 equal factors, it is not a perfect cube. See note above.

$\sqrt[3]{18399744} = \sqrt{(2\cdot2\cdot2\cdot3\cdot11)\,(2\cdot2\cdot2\cdot3\cdot11)\,(2\cdot2\cdot2\cdot3\cdot11)} = 2\cdot2\cdot2\cdot3\cdot11 = 264.$

$\sqrt{4741632} = \sqrt[3]{(2\cdot2\cdot2\cdot3\cdot7)\ (2\cdot2\cdot2\cdot3\cdot7)\ (2\cdot2\cdot2\cdot3\cdot7)} = 2\cdot2\cdot2\cdot3\cdot7 = 168.$

(*127.*) RULE.—*To extract the square root of a perfect square*, resolve the power into two groups of equal factors: the product of the factors in one of these groups is the square root.

To extract the cube root of a perfect cube, resolve the number into three groups of equal factors: the product of the factors in one of these groups is the cube root.

(6.) $\sqrt[5]{5507317760000} =$

$\sqrt{(10\cdot8\cdot7)(10\cdot8\cdot7)(10\cdot8\cdot7)(10\cdot8\cdot7)(10\cdot8\cdot7)} = 10\cdot8\cdot7 = 560.$

Separate the number into 5 groups of equal factors: the product of the factors constituting one of these groups is the 5th root.

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(4.)

$\dot{7}8\dot{9}\dot{3}$ (88.84+
64
168) 1493
1344
1768) 149.00
141 44
17764) 7 5600
7 1056
4544

$4\dot{5}0\dot{0}3\dot{4}$ (670.84+
36
127) 900
889
13408) 1134.00
1072 64
134164) 61 3600
53 6656
7 6944

$\dot{5}8\dot{2}7\dot{1}$ (241.39+
4
44) 182
176
481) 671
481
4823) 190.00
144 69
48269) 45 3100
43 4421
1 8679

$1\dot{1}$ (3.316+
9
63) 2.00
1 89
661) 1100
661
6626) 43900
39756
4144

$\dot{1}1\dot{5}$ (10.723+
1
207) 15.00
14 49
2142) 5100
4284
21443) 81600
64329

$.\dot{3}0$ (.547+
25
104) 500
416
1087) 8400
7609
791

$$\begin{array}{r|l} .44\dot{3}0 & (.665+ \\ 36 & \\ \hline 126) & 830 \\ & 756 \\ \hline 1325) & 7400 \\ & 6625 \\ \hline & 775 \end{array} \qquad \begin{array}{r|l} 64.\dot{8}0 & (8.049+ \\ 64 & \\ \hline 1604) & .8000 \\ & 6416 \\ \hline 16089) & 158400 \\ & 144801 \\ \hline & 13599 \end{array}$$

$$\begin{array}{r|l} 37.62 & (6.133+ \\ 36 & \\ \hline 121) & 1.62 \\ & 121 \\ \hline 1223) & 4100 \\ & 3669 \\ \hline 12263) & 43100 \\ & 36789 \\ \hline & 6311 \end{array} \qquad \begin{array}{r|l} .\dot{0}4\dot{6}0 & (.2145- \\ .04 & \\ \hline 41) & 60 \\ & 41 \\ \hline 424) & 1900 \\ & 1696 \\ \hline 4285) & 20400 \\ & 21425 \\ \hline & \end{array}$$

$$\begin{array}{r|l} \dot{1}.\dot{6}0 &)1.264+ \\ 1 & \\ \hline 22) & .60 \\ & 44 \\ \hline 246) & 1600 \\ & 1476 \\ \hline 2524) & 12400 \\ & 10096 \\ \hline & 2304 \end{array}$$

(5.) $\sqrt{\frac{25}{81}} = \frac{\sqrt{25}}{\sqrt{81}} = \frac{5}{9}$: $\sqrt{\frac{16}{121}} = \frac{\sqrt{16}}{\sqrt{121}} = \frac{4}{11}$: $\sqrt{\frac{49}{169}} = \frac{\sqrt{49}}{\sqrt{169}} = \frac{7}{13}$: $\sqrt{\frac{64}{225}} = \frac{\sqrt{64}}{\sqrt{225}} = \frac{8}{15}$: $\sqrt{\frac{289}{784}} = \frac{\sqrt{289}}{\sqrt{784}} = \frac{17}{28}$.

(6.) $\sqrt{\frac{3}{7}} = \sqrt{.42857142+} = .6546+$:

$\sqrt{\frac{5}{6}} = \sqrt{.833333+} = .91+$:

$\sqrt{\frac{3}{8}} = \sqrt{.375} = .61+$:

$\sqrt{5\frac{1}{2}} = \sqrt{5.5} = 2.34+$:

$\sqrt{7\frac{2}{3}} = \sqrt{7.6666+} = 2.76+$:

$\sqrt{10\frac{5}{9}} = \sqrt{10.5555+} = 3.24+$:

(7.) $\sqrt{\frac{2}{7}} = \sqrt{\frac{2\times7}{7\times7}} = \frac{\sqrt{14}}{7} = \frac{3.741+}{7} = .534+.$

$\sqrt{\frac{5}{8}} = \sqrt{\frac{5\times2}{8\times2}} = \frac{\sqrt{10}}{\sqrt{16}} = \frac{3.16+}{4} = .79+.$

$\sqrt{\frac{7}{24}} = \sqrt{\frac{7\times6}{24\times6}} = \frac{\sqrt{42}}{\sqrt{144}} = \frac{6.48+}{12} = .54+.$

$\sqrt{\frac{1}{12}} = \sqrt{\frac{1\times3}{12\times3}} = \frac{\sqrt{3}}{\sqrt{36}} = \frac{1.732+}{6} = .288+.$

$\sqrt{\frac{1}{3}} = \sqrt{\frac{1\times3}{3\times3}} = \frac{\sqrt{3}}{\sqrt{9}} = \frac{1.732+}{3} = .577+.$

$\sqrt{\frac{11}{13}} = \sqrt{\frac{11\times13}{13\times13}} = \frac{\sqrt{143}}{\sqrt{169}} = \frac{11.95+}{13} = .92-.$

(*129.*) See ELEMENTS OF ARITHMETIC (*289*) for the *Rule.*

DEM.—The reasons for pointing off as we do, are, *first*, that it shows where the square of the highest order in the root is to be found (*94*); then where the square of the two highest orders falls; then of the three, etc. Thus the

square of the highest order in the root is found in the left-hand period; the square of the two highest orders, in the two left-hand periods, etc.; and, *second*, as a consequence of this we never need bring down and consider but one period at a time.

The reason for taking as a *Trial Divisor* 2 times the root already found is, that we are seeking a binomial, one part of which is the root already found, and the other part the figure we are seeking, and that the square of this binomial is the greatest square in the period last brought down and the preceding ones. But of this square we have previously removed the square of the root already found; hence the remainder we are at any time dealing with contains 2 times the root already found (regarded as tens) multiplied by the next figure of the root, + the square of this next figure: or, neglecting this square for an approximation, as it forms an inconsiderable part of the remainder, we may treat the remainder as 2 times the root already found (regarded as tens) multiplied by the next figure of the root. Hence if we divide this remainder by 2 times the root already found (regarded as tens), we shall have, approximately, the next figure of the root.

The *True Divisor* is found by adding to the trial divisor the new figure of the root, since the remainder really contains 2 times the root already found (regarded as tens) + the next figure of the root.

An illustration with letters, as in the text-book, will aid in bringing out the significance of the demonstration.

CUBE ROOT.

PAGES 86-88.

(3.)

$$\begin{array}{rl} & 224755712\,(608 \\ & 216 \\ \hline 3\times(600)^2 = 1080000 & \mid 8755712 \\ 14400 & \mid \\ 64 & \mid \\ \hline 1094464 & \mid 8755712 \end{array}$$

(4.)

$$\begin{array}{rl} & 57856432.163200\,(386.68+ \\ & 27 \\ \hline 3\times(30)^2 = 2700 & \mid 30856 \\ 3\times30\times8 = 720 & \mid \\ 8^2 = 64 & \mid \\ \hline 3484 & \mid 27872 \\ \hline 3\times(380)^2 = 439200 & \mid 2984432 \\ 3\times380\times6 = 6840 & \mid \\ 6^2 = 36 & \mid \\ \hline 446076 & \mid 2676456 \\ \hline 3\times(3860)^2 = 44698800 & \mid 307976.163 \\ 3\times3860\times6 = 69480 & \mid \\ 6^2 = 36 & \mid \\ \hline 44768316 & \mid 268609896 \\ \hline 3\times(38660)^2 = 4483786800 & \mid 39366267200 \\ 3\times38660\times8 = 927840 & \mid \\ 8^2 = 64 & \mid \\ \hline 4484714704 & \mid 35877717632 \\ \hline & 3488549568 \end{array}$$

(5.)

$\dot{1}\dot{1}.$ (2.223+

	8
1200	3.000
120	
4	
1324	2.648
145200	352000
1320	
4	
146524	293048
14785200	58952000
19980	
9	
14805189	44415567
	14536433

$\dot{7}.\dot{6}40$ (1.969+

	1
300	6.640
270	
81	
651	5859
108300	781000
3420	
36	
111756	670536
11524800	110464000
52920	
81	
11577801	104200209
	6263791

$\dot{2}.\dot{7}00$ (1.392+

	1
300	1.700
90	
9	
399	1197
50700	503000
3510	
81	
54291	488619
5796300	14381000
8340	
4	
5804644	11609288
	2771712

200 (5.84+

	125
7500	75.000
1200	
64	
8764	70112
1009200	4888000
6960	
16	
1016176	4064704
	823296

$$\dot{4}\dot{8}5\dot{6}4\dot{3}\dot{2}7\dot{6}\dot{1}\dot{0}0\dot{5}\ (\ 16934.50+$$

	1
300	3856
180	
36	
516	3096
76800	760432
4320	
81	
81201	730809
8568300	29623761
15210	
9	
8583519	25750557
859874700	3873204005
203160	
16	
860077876	3440311504
86028106800	432892501000
2540100	
25	
86030646925	430153234625
8603318707500	2739266375000

$\sqrt[3]{200} = 5.848+$: $\sqrt[3]{178600} = 56.315+$. $\sqrt[3]{3400002.75} = 150.36+$.

(6.) $\sqrt[3]{\frac{27}{125}} = \frac{\sqrt[3]{27}}{\sqrt[3]{125}} = \frac{3}{5}$: $\sqrt[3]{\frac{343}{1331}} = \frac{\sqrt[3]{343}}{\sqrt[3]{1331}} = \frac{7}{11}$: $\sqrt[3]{\frac{8}{2197}} = \frac{\sqrt[3]{8}}{\sqrt[3]{2197}} = \frac{2}{13}$: $\sqrt[3]{\frac{53441}{224755712}} = \frac{\sqrt[3]{53441}}{\sqrt[3]{224755712}} = \frac{37.666+}{608} = .06195+$: $\sqrt[3]{\frac{1}{8}} = \frac{\sqrt[3]{1}}{\sqrt[3]{8}} = \frac{1}{2}$.

(7.) $\sqrt[3]{\frac{3}{5}} = \sqrt[3]{\frac{3\times5\times5}{5\times5\times5}} = \frac{\sqrt[3]{75}}{5} = \frac{4.2171+}{5} =$ $.8434+$: $\sqrt[3]{\frac{1}{2}} = \frac{\sqrt[3]{4}}{2} = \frac{1.587+}{2} = .793+$: $\sqrt[3]{\frac{6}{11}}$ $= \frac{\sqrt[3]{726}}{11} = .817+.$

HIGHER ROOTS.

PAGES 89, 90.

(*131.*) The 6th root is the square root of the 3d root, since if a number is resolved first into 3 equal factors, and then each of these factors resolved into 2 equal factors, the entire number will be resolved into 6 equal factors, one of which is the 6th root.

(1.) $\sqrt[6]{279841} = \sqrt{\sqrt{279841}} = \sqrt{529} = 23$:
$\sqrt[6]{7643256} = \sqrt{\sqrt{7643256}} = \sqrt{2764.64+} = 52.57+$:
$\sqrt[4]{8.5} = \sqrt{\sqrt{8.5}} = \sqrt{2.9154+} = 1.7+$:
$\sqrt[4]{\frac{16}{625}} = \sqrt{\frac{4}{25}} = \frac{2}{5}.$

(2.) $\sqrt[6]{601692057} = \sqrt[3]{\sqrt{601692057}} = \sqrt[3]{24529.412+}$ $= 29.05+$: $\sqrt[6]{825437502} = \sqrt{\sqrt[3]{825437502}} = \sqrt{938.05+}$ $= 30.62+$: $\sqrt[6]{53.2} = \sqrt[3]{\sqrt{53.2}} = \sqrt[3]{7.293833+} =$ $1.93+$: $\sqrt[6]{\frac{64}{16807}} = \sqrt[6]{\frac{64\times7}{16807\times7}} = \frac{\sqrt[6]{448}}{7} =$ $\frac{2.766+}{7} = .395+.$

In solving the last, observe that $16807 = 7^6$; or we may extract the 6th root of 64, obtaining 2; and then get the 6th root of 16807 approximately. The latter is 5.06+; hence we have $\frac{2}{5.06+} = .395+$.

(*133.*) RULE.—*To extract the fifth Root of a number.*

1. Point off the number into periods by placing a mark over the units, and over each fifth figure therefrom, making decimal periods full by annexing 0s.*

2. Take the 5th root of the highest 5th power in the left-hand period as the first figure in the root. Subtract this highest fifth power from the left-hand period, and to the remainder annex the next period, forming a dividend.

3. Take 5 times the root already found (regarded as tens) as a *Trial Divisor*, by which divide the dividend, thus obtaining (approximately) the next figure in the root. Complete the divisor by adding to the trial divisor 10 times the product of the 3d power of the root already found (regarded as tens) into the new figure of the root, also 10 times the square of the root already found (regarded as tens) into the square of the new root figure, also 5 times the root already found (regarded as tens) into the 3d power of the new root figure, and the 4th power of the new root figure. Multiply this *True Divisor* by the last root figure, subtracting the product from the last dividend, and to the remainder annex the next period, forming a new dividend.

4. Repeat the process described in the last paragraph till all the periods have been brought down; and, if there is then a remainder, annex full periods of 0s and continue the process as long as desirable.

* The reason for this pointing is exactly analogous to that for pointing in the square and cube roots. So also in any other root. The student may elaborate it if thought desirable.

(4.) $(x+y)^5 = x^5 + 5x^4y + 10x^3y^2 + 10x^2y^3 + 5xy^4 + y^5 = x^5 + (5x^4 + 10x^3y + 10x^2y^2 + 5xy^3 + y^4)y.$

$22643\overline{52}\overline{07}\overline{11}\overline{94}\overline{3}\,(743$

$x^5 = 7^5 = 16807$

$5\,(x \text{ tens})^4 = 5\,(70)^4 = 120050000$	583652671
$10\,(x \text{ tens})^3y = 10\,(70)^3 4 = 13720000$	
$10\,(x \text{ tens})^2y^2 = 10\,(70)^2 4^2 = 784000$	
$5\,(x \text{ tens})\,y^3 = 5\,(70)\,4^3 = 22400$	
$y^4 = 4^4 = 256$	
$5x^4 + 10x^3y + 10x^2y^2 + 5xy^3 + y^4 = 134576656$	$538306624 = (5x^4 + 10x^3y + 10x^2y^2 + 5xy^3 + y^4)y$
$5\,(x \text{ tens})^4 = 5\,(740)^4 = 1499328800000$	4534604711943
$10\,(x \text{ tens})^3y = 10\,(740)^3 3 = 12156720000$	
$10\,(x \text{ tens})^2y^2 = 10\,(740)^2 3^2 = 49284000$	
$5\,(x \text{ tens})\,y^3 = 5\,(740)\,3^3 = 99900$	
$y^4 = 3^4 = 81$	
$5x^4 + 10x^3y + 10x^2y^2 + 5xy^3 + y^4 = 1511584903981$	$4534604711943 = (5x^4 + 10x^3y + 10x^2y^2 + 5xy^3 + y^4)y.$

(5.) $(x+y)^4 = x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4 = x^4 + (4x^3 + 6x^2y + 4xy^2 + y^3)y.$

The *Trial Divisor* is $4x^3$, and the *True Divisor* is the entire quantity in the parenthesis, x being the part of the root found (regarded as tens) and y the figure next sought. Hence we have the following

RULE.—*To extract the 4th root.*

1. Point off the number into periods by placing a mark over units, and over each 4th figure therefrom, making decimal periods full by annexing 0s.

2. Take the 4th root of the highest 4th power in the left-hand period as the first figure in the root. Subtract this highest 4th power from the left-hand period, and to the remainder annex the next period, forming a dividend.

3. Take 4 times the 3d power of the part of the root already found (regarded as tens), as a *Trial Divisor*, by which divide the dividend, thus obtaining (approximately) the next figure in the root. Complete the divisor by adding to the trial divisor 6 times the product of the square of the root already found (regarded as tens) into the new root figure, also 4 times the root already found (regarded as tens) into the square of the new root figure, and the cube of the new root figure. Multiply this *True Divisor* by the last root figure, subtracting the product from the last dividend, and to the remainder annex the next period.

4. (Same as in other rules.)

	304758098401 (743
	2401
1372000	6465809
117600	
4480	
64	
1494144	5976576
1620896000	4892338401
9856800	
26640	
27	
1630779467	4892338401

PROPERTIES OF NUMBERS

CASTING OUT THE NINES.

PAGES 92–96.

The solution of these examples is so simple as to need no key. We will give an example of the practical process in each case.

Ex. 4, p. 94.

78026
34826
8276
54073
8176
183377

Rejecting the 9s from the numbers added, we say, "6 and 2 are 8, and 8 are 16; 1 and 6 are 7, and 7 are 14; 4 and 1 are 5; 5 and 6 are 11; 1 and 1 are 2; 2 and 2 are 4; 4 and 8 are 12; 1 and 2 are 3; 3 and 4 are 7, and 3 are 10; 1 and 0 are 1; 1 and 6 are 7, and 7 are 14; 1 and 4 are 5;* 7, 15, 6 (1 + 5), 0, 11, 2 (1+1), 7, 13, 4, 11, 2, 3, 11, 2. Thus the excess of 9s in the several numbers added is 2. In the sum we have 14, 5 (1 + 4), 8, 11, 2, 10, 1, 2, giving the excess 2, the same as in the numbers. Hence the addition is probably correct.

In applying this process the following suggestions should be observed:

1. When any sum is exactly 9 reject it, and omit in the process any pair of digits whose sum is 9.

2. When a sum is obtained greater than 9, as 15, instead of subtracting 9 from 15 to find the excess, add the digits.

3. Carry the excess from one of the numbers added to the next, and so on throughout.

(3.) p. 95. As an example in Multiplication take 615.37 × 1.5.

* Up to this point the full form is given, in order to show the method of procedure; also, in order that the process may be the more readily traced, we have proceeded from right to left. See *Practical Suggestions*, p. 93, and (*138*) in text-book.

615.37 Excess of 9s, 4
1.5 Excess of 9s, 6
307685 Prod. of Excesses, 24: Excess in this Prod., 6.
61537
923055 Excess in Prod., 6.

As the excess in the product is equal to the excess in the product of the excesses in the factors, the work is probably correct.

As an example in Subtraction take 5804726 — 325891.

5804726 Excess of 9s, 5.
325891
5478835 } Excess of 9s in both, 5.

Hence the work is probably correct.

(6.) p. 96.

729386|731
6579 997$\frac{419}{731}$
7148
6579
5696
5117
579

The excess of 9s in the divisor is 2, and in the quotient 7. The excess in the product of these excesses is (1 + 4) 5. Adding this to the digits of the remainder, and rejecting the 9s, we have 8, which is also the excess in the dividend; hence the work is probably correct.

(***148.***) DEM.—Any number may be considered as made up of two parts, *first*, as many 10s as are represented by the figures exclusive of the units figure, and, *second*, of the units. Now, since a number is divisible by a factor of any factor contained in it, 5, a factor of 10, is a divisor of the first part. Hence, if the units figure is 5 or 0, the entire number is divisible by 5, and not otherwise.

(***149.***) DEM.—*First*, any number contains a factor 3 the sum of whose digits is divisible by 3 (Remark under ***146***). *Second*, as removing this factor neither introduces nor removes a factor 2, if the number contained such a factor,

i. e., was *even*, before the division by 3, it will contain it after. Hence an even number the sum of whose digits is divisible by 3 is divisible by 2×3, or 6, and no other number is so divisible.

(***151.***) DEM.—Any number may be considered as made up of as many *thousands* as are represented by the figures exclusive of the 3 right-hand ones, + the number represented by these 3 figures. Now the former of these parts is divisible by 8, since 1000, one of its factors, is so divisible. Hence if the number represented by the last three digits is divisible by 8, the entire number is, and not otherwise.

(***152.***) DEM.—From (***134***) we have

$$\underbrace{999a + 99b + 9c}_{\text{1st part.}} + \underbrace{a + b + c + d}_{\text{2d part.}}$$

Now the first part is an exact number of 9s; hence, if the 2d part, which is the sum of the digits, is divisible by 9, the entire number is, and not otherwise.

(***153.***) DEM.—As any number is as many 10s as are represented by the figures exclusive of the right-hand one, + the number represented by the right-hand figure, if this is 0 the number is an exact number of 10s, and not otherwise.

(***156.***) DEM.—If the sum of the digits is divisible by 3 the entire number is, and not otherwise (***146***, 2). But rejecting this factor neither introduces nor rejects a factor 4; hence if the number was divisible by 4 before the factor 3 was rejected, it will be after. Now if the number represented by the last two figures is divisible by 4 the entire number is, and not otherwise (***147***). Hence any number whose last two figures represent a number divisible by 4, and the sum of whose digits is divisible by 3, is divisible by 4×3, or 12.

PAGE 100.

(2.) By 2 since it is an even number, the last digit being even.—By 4 since 56 is divisible by 4.—By 7 since the unit's digit, + 3 times the tens, $+3^2$ times the hundreds, + 3^3 times the thousands, + etc., or 6 + 15 + 18 + 108 + 486 + 1215 = 1848, is divisible by 7. Or, more simply, since 56 (ten thousands) + 42 (hundreds) + 56 is divisible by 7.—By 8 since 256 is so divisible.—By 11 since the sum of the digits in the odd places (6 + 2 + 6 = 14), is equal to the sum of those in the even places (5 + 4 + 5 = 14).

(3.) By 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, *i. e.*, by all the numbers inclusive.

GREATEST COMMON DIVISOR.

PAGE 102.

(*167.*) RULE.—*To find the G. C. D. of two or more numbers*, resolve them into their prime factors and take the product of all the common factors.

(2.) $3252 = 2 \times 2 \times 3 \times 271,$
$4248 = 2 \times 2 \times 2 \times 531.$

Now as 2×2 is the product of all the common factors it is the G. C. D.

$506 = (2 \times 11) \times 23$
$308 = (2 \times 11) \times 2 \cdot 7$
$\therefore 2 \times 11 = 22$ is the G. C. D.

$110 = (2 \times 5) \times 11$
$140 = (2 \times 5) \times 2 \times 7$
$680 = (2 \times 5) \times 2 \times 2 \times 17$
$\therefore 2 \times 5 = 10$ is the G. C. D.

$56 = (2 \times 2 \times 7) \times 2$
$84 = (2 \times 2 \times 7) \times 3$
$140 = (2 \times 2 \times 7) \times 5$
$168 = (2 \times 2 \times 7) \times 2 \times 3$
$\therefore 2 \times 2 \times 7 = 28$ is the G. C. D.

PAGES 104–106.

(2.)

156479	100259	1
100259	56220	1
56220	44039	1
44039	36543	3
12181	7496	1
7496	4685	1
4685	2811	1
2811	1874	1
1874	937	2
1874		

∴ 937 is the G. C. D.

4248	3252	1
3252	2988	3
996	264	3
792	204	1
204	60	3
180	48	2
24	12	2
24		

∴ 12 is G. C. D.

(5.)

1978	1311	805	1, 3, 1.
1518	805	506	
460	506	299	1, 2, 6.
414	299	276	
46	207	23	9, 2.
46	207		

∴ 23 is G. C. D.

1347	1134	837	1, 5, 12.
1134	1065	69	
213	69	144	11, 71.
213	66	138	
	3	6	2
		6	

∴ 3 is G. C. D.

3391	2113	1481	1, 2, 5.
3160	1481	1264	
231	632	217	1, 45, 15.
217	630	210	
14	2	7	

∴ As 2 and 7 are prime to each other, there is no G. C. D. above unity.

2886	2220	1998	1554	2.*
1443	1110	999	777	3.*
481	370	333	259	1,5,6.
444	370	259	259	7.
37		74		2.
		74		

∴ $2 \times 3 \times 37 = 222$ is the G. C. D.

MULTIPLES.

PAGES 108–110.

(1.) An infinite number, since any multiple multiplied by any whole number gives a multiple. $5 \times 7 \times 6 \times 3 = 630$ is a C. M., and $5 \times 7 \times 6 = 210$ is the L. C. M.

(2.) $11 \times 13 = 143$, L. C. M.: $17 \times 23 \times 5 = 1955$, L. C. M. Since the numbers are prime to each other their product is their L. C. M.

(*178.*) See ELEMENTS OF ARITHMETIC (*120*).

(5.)

$14 = 2 \times 7$
$28 = 2 \times 7 \times 2$
$21 = 3 \times 7$
$42 = 3 \times 7 \times 2$

42 lacks one factor 2 of containing all the factors of 28; and as it contains all the factors of any other of the numbers, $42 \times 2 = 84$ is the L. C. M.

(6.)

$40 = 2 \cdot 5 \cdot 2 \cdot 2$
$60 = 2 \cdot 5 \cdot 2 \cdot 3$
$90 = 2 \cdot 5 \cdot 3 \cdot 3$
$80 = 2 \cdot 5 \cdot 2 \cdot 2$
$30 = 2 \cdot 5 \cdot 3$

$\therefore 90 \times 2 \times 2 = 360$ is the L. C. M.

(7.)

$14 = 2 \cdot 7$
$27 = 3 \cdot 3 \cdot 3$
$54 = 3 \cdot 3 \cdot 3 \cdot 2$
$63 = 3 \cdot 3 \cdot 7$
$81 = 3 \cdot 3 \cdot 3 \cdot 3$

$\therefore 81 \times 7 \times 2 = 1134$ is the L. C. M.

(8.)

$15 = 3 \cdot 5$
$42 = 3 \cdot 2 \cdot 7$
$72 = 2 \cdot 2 \cdot 2 \cdot 3 \cdot 3$
$81 = 3 \cdot 3 \cdot 3 \cdot 3$

$\therefore 81 \cdot 2 \cdot 2 \cdot 2 \cdot 7 \cdot 5 = 22680$ is the L. C. M.

(9.)

$35 = 7 \cdot 5$
$45 = 3 \cdot 3 \cdot 5$
$81 = 3 \cdot 3 \cdot 3 \cdot 3$
$96 = 3 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2$

$\therefore 96 \cdot 3 \cdot 3 \cdot 3 \cdot 5 \cdot 7 = 90720$ is the L. C. M.

(10.)

$18 = 2 \cdot 3 \cdot 3$
$35 = 5 \cdot 7$
$50 = 2 \cdot 5 \cdot 5$
$60 = 3 \cdot 2 \cdot 2 \cdot 5$
$144 = 3 \cdot 3 \cdot 2 \cdot 2 \cdot 2 \cdot 2$

$\therefore 144 \cdot 5 \cdot 5 \cdot 7 = 25200$ is the L. C. M.

(11.) 2 and 3 are prime; $21 = 3 \cdot 7$; $14 = 2 \cdot 7$; $63 = 7 \cdot 3 \cdot 3$. ∴ The L. C. M. is $63 \cdot 2 = 126$.

(12.) 41, 37, 11, 23, and 71 being all prime, their L. C. M. is $41 \times 37 \times 11 \times 23 \times 71 = 27249871$.

(13.) $6 = 2 \cdot 3$; $36 = 3 \cdot 3 \cdot 2 \cdot 2$; $24 = 2 \cdot 2 \cdot 2 \cdot 3$; $48 = 3 \cdot 2 \cdot 2 \cdot 2 \cdot 2$; and 2 is prime. ∴ The L. C. M. is $48 \cdot 3 = 144$.

(14.) $20 = 2 \cdot 2 \cdot 5$; $60 = 3 \cdot 5 \cdot 2 \cdot 2$; $15 = 3 \cdot 5$; $165 = 3 \cdot 11 \cdot 5$; $63 = 3 \cdot 3 \cdot 7$; and $210 = 2 \cdot 5 \cdot 3 \cdot 7$. ∴ The L. C. M. is $210 \cdot 3 \cdot 11 \cdot 2 = 13860$.

(15.)

54479	35741	1, 1.
35741	18738	
18738	17003	1, 9.
17003	15615	
1735	1388	1, 4.
1388	1388	
347		

Now $35741 \div 347 = 103$. Hence $54479 \times 103 = 5611337$ is the G. C. D. of 54479 and 35741.

(16.)

88409	63269	31861	1, 2, 1.
62816	31861	31408	
25593	31408	453	69, 3, 169.
151	2718	453	
1049	4228		
906	4077		
1433	151		
1359			
74			

151 is a common factor of 63269 and 31861; but 88409 is prime to each of the others. $63269 = 419 \times 151$; and $31861 = 211 \times 151$. ∴ The L. C. M. of the three numbers is $88409 \times 63269 \times 211 = 1180238843431$.

SIMPLE EQUATIONS.

PAGES 113, 114.

(3.) x representing the price per bushel, he received for the first lot $13x$, and for the 2d $17x$;

hence, $17x - 13x = 36$.

Uniting terms, $4x = 36$.

Dividing each member by 4, $x = 9$, the price per bushel.

(4.) x representing the price per bushel, he received ax for the first lot, and bx for the 2d;

hence, $bx - ax = c$.

Uniting terms, $(b - a)x = c$.

Dividing each member by $b - a$, $x = \frac{c}{b - a}$, the price per bushel.

(5.) Letting x represent what he spent:

$$3x + x = 40, \quad 4x = 40, \quad x = 10.$$

(6.) Letting x represent what he spent:

$$bx + x = a, \quad (b + 1)x = a, \quad x = \frac{a}{b + 1}.$$

(11.) $x + \frac{x}{3} = 60$. (12.) $x + \frac{x}{a} = b$.

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(4.)

$$7x - 16 = 3x - 4$$
$$7x - 3x = 16 - 4$$
$$4x = 12$$
$$x = 3$$

(5.)

$$4x - 48 = 24 - 2x$$
$$4x + 2x = 48 + 24$$
$$6x = 72$$
$$x = 12$$

(6.)

$$6 - 2x + 10 = 20 - 3x - 2$$
$$3x - 2x = 20 - 2 - 6 - 10$$
$$x = 2$$

(7.)

$$5x + 16 = 3x + 100$$
$$5x - 3x = 100 - 16$$
$$2x = 84$$
$$x = 42$$

(8.)

$$3x + 4 - 80 + x = 22 - 10x$$
$$3x + 10x + x = 22 + 80 - 4$$
$$14x = 98$$
$$x = 7$$

(12.)

$$mx + ab = nx + 4ab$$
$$mx - nx = 4ab - ab$$
$$(m - n)x = 3ab$$
$$x = \frac{3ab}{m - n}$$

(13.)

$$4bx - 5am = 2bx - 10an$$
$$4bx - 2bx = 5am - 10an$$
$$2bx = 5am - 10an$$
$$x = \frac{5am - 10an}{2b}$$

(16.)

$$\tfrac{2}{3}x + 5 = \tfrac{1}{6}x + 2$$
$$9x + 60 = 10x + 24$$
$$9x - 10x = 24 - 60$$
$$-x = -36$$
$$x = 36$$

The last operation may be considered as multiplying by -1.

(17.)

$\frac{x+1}{5} + 2 = \frac{2x-3}{3} - 1$. Multiply each number by 3×5.

$$3x + 3 + 30 = 10x - 15 - 15$$
$$3x - 10x = -15 - 15 - 3 - 30$$
$$-7x = -63.$$ Divide each number by -7.
$$x = 9$$

(18.)

$$\frac{7x}{8} - 5 = \frac{9x}{10} - 8$$
$$35x - 200 = 36x - 320$$
$$35x - 36x = 200 - 320$$
$$-x = -120$$
$$x = 120$$

(19.)

$$\begin{aligned}
\tfrac{1}{3}x + 10 &= \tfrac{1}{5}x + \tfrac{1}{4}x + \tfrac{1}{3}x - 7 \\
30x + 600 &= 20x + 15x + 12x - 420 \\
30x - 20x - 15x - 12x &= -600 - 420 \\
-17x &= -1020 \\
x &= 60
\end{aligned}$$

(20.)

$$\begin{aligned}
56 - \tfrac{1}{4}x &= 48 - \tfrac{1}{8}x \\
448 - 6x &= 384 - 5x \\
5x - 6x &= 384 - 448 \\
-x &= -64 \\
x &= 64
\end{aligned}$$

(21.)

$$\begin{aligned}
25\,(11x - 8) &= 18\,(12x + 2) \\
275x - 200 &= 216x + 36 \\
275x - 216x &= 200 + 36 \\
59x &= 236 \\
x &= 4
\end{aligned}$$

(22.)

$$\begin{aligned}
x(x + 5) &= 100 - 20x + x^2 \\
x^2 + 5x &= 100 - 20x + x^2 \\
20x + 5x &= 100 \\
25x &= 100 \\
x &= 4
\end{aligned}$$

(23.)

$$\begin{aligned}
x - rx &= 100 \\
(1 - r)x &= 100 \\
x &= \frac{100}{1 - r}
\end{aligned}$$

(24.)

$$\begin{aligned}
42 - \frac{3x + 2}{5} &= 8 + 5x \\
210 - 3x - 2 &= 40 + 25x \\
-3x - 25x &= 40 - 210 + 2 \\
-28x &= -168 \\
x &= 6
\end{aligned}$$

(25.)

$$\begin{aligned}
x + \frac{3x - 5}{2} &= 12 - \frac{2x - 4}{3} \\
6x + 9x - 15 &= 72 - 4x + 8 \\
6x + 9x + 4x &= 72 + 8 + 15 \\
19x &= 95 \\
x &= 5
\end{aligned}$$

(26.)

$$\frac{x+1}{3} - \frac{x+4}{5} = 16 - \frac{x+3}{4}$$

$$20x + 20 - 12x - 48 = 960 - 15x - 45$$

$$20x - 12x + 15x = 960 - 45 - 20 + 48$$

$$23x = 943$$

$$x = 41$$

(27.)

$$8x - \frac{11x - 37}{2} = 5 - \frac{2x + 6}{5}$$

$$30x - 55x + 185 = 50 - 4x - 12$$

$$30x - 55x + 4x = 50 - 12 - 185$$

$$-21x = -147$$

$$x = 7$$

(28.)

$$\frac{x-1}{7} = 7 - \frac{23 - x}{5} - \frac{4 + x}{4}$$

$$20x - 20 = 980 - 644 + 28x - 140 - 35x$$

$$20x - 28x + 35x = 980 - 644 - 140 + 20$$

$$27x = 216$$

$$x = 8$$

(29.)

$$\frac{3x + 4}{5} - \frac{7x - 3}{2} = \frac{x - 16}{4}$$

$$12x + 16 - 70x + 30 = 5x - 80$$

$$12x - 70x - 5x = -80 - 16 - 30$$

$$-63x = -126$$

$$x = 2$$

(30.)

$$ax - \frac{a^2 - 3bx}{a} - ab^2 = bx + \frac{6bx - 5a^2}{2a} - \frac{bx + 4a}{4}$$

$$4a^2x - 4a^2 + 12bx - 4a^2b^2 = 4abx + 12bx - 10a^2 - abx - 4a^2$$

$$4a^2x - 4abx + abx = 4a^2b^2 - 10a^2$$

$$4a^2x - 3abx = 4a^2b^2 - 10a^2$$

$$(4a^2 - 3ab)x = 2a^2(2b^2 - 5)$$

$$x = \frac{2a^2(2b^2 - 5)}{4a^2 - 3ab} = \frac{2a(2b^2 - 5)}{4a - 3b}$$

(31.)

$$\frac{x}{a} - \frac{dx}{c} + 3ab = 1$$

$$cx - adx + 3a^2cb = ac$$

$$cx - adx = ac - 3a^2cb$$

$$(c - ad)x = ac(1 - 3ab)$$

$$x = \frac{ac(1 - 3ab)}{c - ad}$$

(32.)

$$\frac{(a - b)x}{2} + \frac{x}{3} = \frac{ab}{4} + a$$

$$6(a - b)x + 4x = 3ab + 12a$$

$$[6(a - b) + 4]x = 3ab + 12a$$

$$x = \frac{3ab + 12a}{6(a - b) + 4} = \frac{3a(b + 4)}{6(a - b) + 4}$$

(33.) Since $x^2 = 4$, extracting the square root of each member, we have $x = \pm 2$.

(34.)

$$3x^2 = 48$$

$$x^2 = 16$$

$$x = \pm 4$$

(35.)

$$6x^2 - 2x^2 = 100$$

$$4x^2 = 100$$

$$x^2 = 25$$

$$x = \pm 5$$

(37.)

$$35 - \frac{x^2 + 50}{5} = x^2 - \frac{x^2 - 10}{3}$$

$$525 - 3x^2 - 150 = 15x^2 - 5x^2 + 50$$

$$5x^2 - 3x^2 - 15x^2 = -525 + 150 + 50$$

$$-13x^2 = -325$$

$$x^2 = 25$$

$$x = \pm 5$$

(38.)

$$\begin{aligned} ax^2 - b &= c \\ ax^2 &= c + b \\ x^2 &= \frac{c+b}{a} \\ x &= \pm\sqrt{\frac{c+b}{a}} \\ &= \pm\sqrt{\frac{a(c+b)}{a^2}} \\ &= \pm\frac{1}{a}\sqrt{a(c+b)} \end{aligned}$$

(39.)

$$\begin{aligned} ax^2 - 5c &= bx^2 - 3c + d \\ ax^2 - bx^2 &= 5c - 3c + d \\ (a-b)x^2 &= 2c + d \\ x^2 &= \frac{2c+d}{a-b} \\ x &= \pm\sqrt{\frac{2c+d}{a-b}} = \pm\sqrt{\frac{(a-b)(2c+d)}{(a-b)^2}} \\ &= \pm\frac{1}{a-b}\sqrt{(a-b)(2c+d)} \end{aligned}$$

PAGES 121–124.

(2.)

$7x - 4y - 3 = 70$ gives $7x - 4y = 73$, . . (1)

and $\frac{1}{3}x + \frac{1}{2}y - 3 = 4$ " $2x + 5y = 70$. . . (2)

From (1) $x = \frac{73+4y}{7}$.

Substituting this in (2), $\frac{146+8y}{7} + 5y = 70$. . . (3)

From (3) $146 + 8y + 35y = 490$; whence $y = 8$.

Substituting this value of y for y in the value of x in terms of y, we have

$$x = \frac{73 + 4\cdot 8}{7} = \frac{73+32}{7} = \frac{105}{7} = 15.$$

(3.)

$\frac{1}{3}x + \frac{1}{2}y = 10$ gives $5x + 3y = 150$, . . . (1)
and $\frac{1}{5}x + \frac{1}{6}y = 11$ " $3x + y = 66$. . . . (2)

From (1) $y = \frac{150 - 5x}{3}$.

Substituting this in (2), we have

$$3x + \frac{150 - 5x}{3} = 66\text{; whence } x = 12.$$

And $y = \frac{150 - 5x}{3} = \frac{150 - 5 \cdot 12}{3} = \frac{150 - 60}{3} = 30.$

(4.)

From $6x + 7y = 33$, $x = \frac{33 - 7y}{6}$; and substituting this in $8x - 3y = 7$ we have $\frac{132 - 28y}{3} - 3y = 7$; whence $y = 3$. Again $x = \frac{33 - 7y}{6} = \frac{33 - 7 \cdot 3}{6} = \frac{33 - 21}{6} = 2.$

(5.)

From $16x - 13y - 1 = 30$, $x = \frac{31 + 13y}{16}$; and substituting this in $24x + 15y - 19 = 200$, we have $\frac{93 + 39y}{2} + 15y = 219$; whence $y = 5$.

Again, $x = \frac{31 + 13y}{16} = \frac{31 + 13 \cdot 5}{16} = \frac{31 + 65}{16} = \frac{96}{16} = 6.$

(6.)

From $5x - 4y = 19$, $x = \frac{19 + 4y}{5}$; and substituting in $4x + 2y = 36$, $\frac{76 + 16y}{5} + 2y = 36$; whence $76 + 16y + 10y = 180$, $26y = 180 - 76$, and $y = 4$.

Hence, $x = \frac{19 + 4y}{5} = \frac{19 + 16}{5} = \frac{35}{5} = 7.$

(7.)

From $ax - by = a^2$, $x = \frac{a^2 + by}{a}$; and substituting in $bx - ay = b^2$, $\frac{a^2b + b^2y}{a} - ay = b^2$; clearing of fractions, $a^2b + b^2y - a^2y = ab^2$; transposing and uniting terms $(b^2 - a^2)y = ab(b - a)$; dividing by $b^2 - a^2$, $y = \frac{ab(b-a)}{b^2-a^2} = \frac{ab(b-a)}{(b+a)(b-a)} = \frac{ab}{b+a}$.

Hence, $x = \frac{a^2+by}{a} = \frac{a^2+\frac{ab^2}{b+a}}{a} = a + \frac{b^2}{b+a} = \frac{ab+a^2+b^2}{b+a}$.

(8.)

Clearing $\frac{x}{b} - \frac{y}{a} = 1$ of fractions, $ax - by = ab$; whence $ax = ab + by$. Substituting this value of ax for ax in $ax + by = c$, we have $ab + by + by = c$; whence $y = \frac{c - ab}{2b}$. Substituting this value of y in $ax = ab + by$, $ax = ab + \frac{c - ab}{2} = \frac{2ab + c - ab}{2} = \frac{ab + c}{2}$; and dividing by a, $x = \frac{ab + c}{2a}$.

(9.)

$ax - by = c$ gives $x = \frac{c + by}{a}$, which substituted in $x + ay = b$, gives $\frac{c + by}{a} + ay = b$; whence $y = \frac{ab - c}{a^2 + b}$.

Hence, $x = \frac{c+by}{a} = \frac{c + \frac{ab^2-bc}{a^2+b}}{a} = \frac{a^2c+bc+ab^2-bc}{a(a^2+b)} = \frac{ac + b^2}{a^2 + b}$.

(10.)

$a(a + x) = b(b + y)$ gives $a^2 + ax = b^2 + by$, and $ax = b^2 + by - a^2$. This substituted in $ax + by = c^2$ gives $b^2 + by - a^2 + by = c^2$; whence $y = \frac{a^2 + c^2 - b^2}{2b}$.

Hence, $ax = b^2 + \frac{a^2 + c^2 - b^2 - 2a^2}{2} = \frac{b^2 - a^2 - c^2}{2}$,

and $x = \frac{b^2 - a^2 + c^2}{2a}$.

(11.)

$ax - by = c$ gives $x = \frac{c + by}{a}$. Substituting this in $x + ay = 3c$, $\frac{c + by}{a} + ay = 3c$; whence $y = \frac{(3a-1)c}{a^2+b}$. Substituting this value of y in $ax - by = c$, we have $ax - \frac{(3a - 1)bc}{a^2 + b} = c$. Transposing and expanding the numerator, $ax = c + \frac{3abc - bc}{a^2 + b} = \frac{a^2c + bc + 3abc - bc}{a^2 + b}$ $= \frac{a^2c + 3abc}{a^2 + b}$. Dividing by a, $x = \frac{ac + 3bc}{a^2 + b} = \frac{c(a + 3b)}{a^2 + b}$.

(12.)

From $p = br$, $b = \frac{p}{r}$. This substituted in $A = b + br$ gives $A = \frac{p}{r} + p$. Again, from $p = br$, $r = \frac{p}{b}$; which substituted in $A = b + br$ gives $A = b + p$. Again, from $A = b + br$, $r = \frac{A - b}{b}$; which substituted in $p = br$ gives $p = A - b$. Again, from $A = b + br$, $b = \frac{A}{1 + r}$; which substituted in $p = br$ gives $p = \frac{Ar}{1 + r}$.

(13.)

Given $i = prt$ (1), and $A = p + i$ (2).

1. Substituting in (2) the value of i from (1), $A = p + prt = p(1 + rt)$.

2. Dividing each member of (1) by pt, $r = \frac{i}{pt}$.

3. Substituting in (2) the value of i from (1), and $A = p + prt$; whence $prt = A - p$, and $r = \frac{A - p}{pt}$.

4. Since $A = p + prt$, $prt = A - p$, and $t = \frac{A - p}{pr}$.

5. Since $i = prt$, $t = \frac{i}{pr}$.

(14.)

Given $l = a + (n - 1)d$, and $s = \left(\frac{a + l}{2}\right)n$.

1. Substituting the value of l as given in (1) in (2), $s = \frac{a + a + (n - 1)d}{2} n = \left(\frac{2a + (n - 1)d}{2}\right)n$.

2. From (1) by transposing and dividing by $n - 1$ we have $d = \frac{l - a}{n - 1}$.

3. (1) gives $(n - 1)\, d = l - a$; whence $n - 1 = \frac{l - a}{d}$, and $n = \frac{l - a}{d} + 1$.

4. From (1) $n = \frac{l - a}{d} + 1$, as we found in (3). Substituting this in (2) and $s = \left(\frac{a + l}{2}\right)\left(\frac{l - a}{d} + 1\right) = \left(\frac{a + l}{2}\right)\left(\frac{l - a + d}{d}\right) = \frac{(l + a)\ (l - a + d)}{2d} = \frac{l^2 - a^2 + d\ (l + a)}{2d} = \frac{l^2 - a^2}{2d} + \frac{l + a}{2}$.

5. From (2) $2s = an + ln$; whence $n = \frac{2s}{a+l}$.

6. From (2) as above $an + ln = 2s$; whence $a = \frac{2s - ln}{n} = \frac{2s}{n} - l$.

7. From (1) $a = l - (n-1)d$. Substituting this value in (2) $s = \left(\frac{l - (n-1)d + l}{2}\right)n = \left(\frac{2l - (n-1)d}{2}\right)n$.

RATIO.

PAGES 126–128.

(1.) $12 : 4 = 3$. $8 : 2 = 4$. $7 : 5 = \frac{7}{5}$. $m : n = \frac{m}{n}$. $x : y = \frac{x}{y}$. In case of the colon* used as a sign of ratio it is the exact equivalent of $\div$.

(2.) In the ratio $\frac{4}{5}$ 4 is antecedent and 5 is consequent. See ***198*** for answers and reasons.

(3.) See ***198***.

(4.) $14 : 2 = 7$. $35 : 7 = 5$. $8 : 4$. $77 : 7$. $120 : 4$.

(5.) $18 : 6 = 3$. $35 : 7 = 5$. $143 : 11 = 13$. $ar : a = r$. $nl : l = n$.

(6.) $18 : 6 = 3$. $5 : 5 = 1$. $5 : 1 = 5$. $ar : a = r$. $a : \frac{a}{r} = r$.

(7.) $\frac{3}{5} = \frac{\frac{3}{5} \times 3 \cdot 7}{\frac{5}{7} \times 3 \cdot 7}$ (See ***198***) $= \frac{11}{15}$. $\frac{4}{5} = \frac{\frac{4}{5} \times 10}{\frac{3}{10} \times 10} = \frac{8}{3} = 2\frac{2}{3}$. $\frac{1\frac{1}{2}}{2\frac{2}{3}} = \frac{1\frac{1}{2} \times 2 \cdot 3}{2\frac{2}{3} \times 2 \cdot 3} = \frac{9}{16}$. $\frac{\frac{a}{b} \times b \cdot y}{\frac{x}{y} \times b \cdot y} = \frac{ay}{bx}$. $\frac{10 \cdot 5}{2 \cdot 25} = \frac{10 \cdot 5 \times 100}{2 \cdot 23 \times 100} = \frac{1050}{225} = 4\frac{2}{3}$. $\frac{5}{.05} = \frac{5 \times 100}{.05 \times 100} = \frac{500}{5} = 100$. $\frac{.05}{5} = \frac{.05 \times 100}{5 \times 100} = \frac{5}{500} = \frac{1}{100}$.

* This sign is here sometimes used for brevity.

(8.) $\frac{1\ ft.}{1\ yd.} = \frac{1}{1 \times 3} = \frac{1}{3}$. $\frac{1\ ft.}{1\ in.} = \frac{1 \times 12}{1} = 12$. $\frac{1\ ft.}{1\ rd.} = \frac{1}{1 \times 16\frac{1}{2}} = \frac{2}{33}$. $\frac{2\ gal.}{3\ qt.} = \frac{2 \times 4}{3} = \frac{8}{3} = 2\frac{2}{3}$. $\frac{\$5}{4\ fr.} = \frac{500}{4 \times 19.3} = \frac{500}{77.2} = \frac{5000}{772} = 6\frac{368}{193}$. $\frac{1\ M.}{1\ fr.} = \frac{1 \times 238}{1 \times 193} = \frac{238}{193} = 1\frac{45}{193}$. $\frac{1\ lb.\ T.}{1\ lb.\ Av.} = \frac{1 \times 5760}{1 \times 7000} = \frac{5760}{7000} = \frac{144}{175}$.

(9.) $\frac{12}{x} = 3$; $\therefore x = 4$. $\frac{x}{5} = 2$; $\therefore x = 10$. $\frac{x}{z} = r$; $\therefore z = \frac{x}{r}$. $\frac{z}{m} = x$; $\therefore z = mx$.

(10.) $\frac{5}{6} \times \frac{3}{4} \times \frac{4}{5} \times \frac{7}{8} \times \frac{12}{14} = \frac{3}{16}$.

(11.) $\frac{a}{b} \times \frac{bx}{a^2y} \times \frac{xy}{y^3} \times \frac{a^2m}{n} = \frac{amx^2}{ny^3}$.

PROPORTION.

Pages 129–132.

(2.) $\frac{45}{\$15} :: \frac{\$28.12\frac{1}{2}}{x}$; $\therefore x = \$132.50$.

(3.) $\frac{\$14}{196} = \frac{\$238.5}{x}$; $\therefore x = \$219.375$.

(5.) The greater the number of days the less the number of men; hence, $\frac{16\frac{2}{3}}{6\frac{2}{3}} = \frac{81}{x}$; $\therefore x = 42$.

(6.) $\frac{4\frac{2}{3}}{81} = \frac{6\frac{2}{3}}{x}$; $\therefore x = 16\frac{2}{7}$.

(8.) $\frac{\frac{5}{6}}{\frac{5}{12}} = \frac{5.25}{x}$, or $2 = \frac{5.25}{x}$; $\therefore x = 2.625$.

(9.) $\frac{6\frac{2}{3}}{12\frac{2}{7}} = \frac{42}{x}$, or $\frac{\frac{20}{3}}{\frac{90}{7}} = \frac{\frac{2}{3}}{\frac{9}{7}} = \frac{14}{27} = \frac{42}{x}$, or $\frac{1}{27} = \frac{3}{x}$; $\therefore x = 81$.

(10.) $\frac{21}{42} = \frac{12\frac{6}{7}}{x}$, $\frac{21}{6} = \frac{90}{x}$, $\frac{7}{6} = \frac{10}{x}$, $\frac{7}{2} = \frac{10}{x}$, $3 = \frac{20}{x}$, $x = \frac{20}{3} = 6\frac{2}{3}$. This and the preceding solutions are given in this form to suggest the utility of such solutions in familiarizing the pupil with the various reductions of a ratio.

(11.) $\frac{4}{52} = \frac{27}{x}$; $\frac{1}{13} = \frac{27}{x}$; $\therefore x = 13 \times 27 = 351$.

(12.) $\frac{5}{1} = \frac{4}{x}$; $\frac{10}{3} = \frac{4}{x}$; $\frac{5}{6} = \frac{1}{x}$; $x = \frac{6}{5} = 1.20$.

(13.) $\frac{3\frac{3}{4}}{3\frac{1}{8}} = \frac{8}{x}$; $\therefore x = 7\frac{1}{2}$.

(14.) $\frac{44\frac{1}{8}}{33\frac{1}{4}} = \frac{9\frac{1}{2}}{x}$; $\therefore x = 7\frac{1}{8}$.

(16.) $\frac{10\ M.}{\$50} = \frac{1\ stere}{x\ cords}$, or $\frac{10 \times .238}{50} = \frac{.2759}{x}$; $x =$ 5 *cd.* 102— *cu. ft.*

(17.) $\frac{7}{256} :: \frac{1}{x}$; $x = 34$.

(*206.*)

(1.) $\sqrt{4.9} = \sqrt{36} = 6$. $\sqrt{27 \times 3} = \sqrt{81} = 9$. $\sqrt{5 \times 11} = \sqrt{55} = 7.416+$. $\sqrt{\frac{2}{3} \times \frac{1}{2}} = \sqrt{\frac{1}{3}} = \frac{\sqrt{3}}{3} = .5773+$.

(2.) $\frac{7}{4} :: \frac{4}{x}$; $\therefore x = 2\frac{2}{7}$. $\frac{5}{12} :: \frac{12}{x}$; $x = 18$. $\frac{12}{8} :: \frac{8}{x}$; $x = 5\frac{1}{3}$. $\frac{4}{6} :: \frac{6}{x}$; $\therefore x = 9$. $\frac{6}{4} :: \frac{4}{x}$; $\therefore x = 2\frac{2}{3}$. $\frac{3}{4\frac{1}{2}} :: \frac{4}{x}$; $\therefore x = \frac{14}{15}$.

COMPOUND PROPORTION.

PAGE **134.**

For this cumbrous process there is not the slightest use in mathematics, and the author feels like apologizing for having given it even so small a place in the text. It is far

better that all such problems should be analyzed in detail. Nevertheless, if any one will use the process of compounding, *it should be shown clearly why the ratios are compounded.* It is solely for the presentation of this thought that the subject is allowed any place. The ordinary mechanical way of treating this subject is worse than useless; it is positively vicious, in an educational course.

(6.)

	hr.	*da.*	*Length.*	*Width.*	*Depth.*	*Men.*
Given case,	8	18	95	12	9	24
Required case,	12	24	380	9	6	x

Let us take the conditions in their order:

1st. If 24 men are required working 8 *hr.* per day, how many will be required when they work 12 *hr.*? The single point we are to consider is whether the number of men required will be in the ratio of the number of hours worked per day, or in the inverse ratio. Evidently in the inverse ratio, *i. e.*, the more hours they work per day, the less men it will take. Hence, *if this were the only condition,* we should have $\frac{12}{8} = \frac{24}{x}$.

2d. But the required men are to work more days. Now will this make the ratio of the given number of men to the required number $\left(i.\ e., \frac{24}{x}\right)$ greater or less? Evidently it will make the number of men required *less,* and as it makes x less, it makes the ratio $\frac{24}{x}$ *greater.* But how much? If there were to be *twice* as many days, it would make the number of men only $\frac{1}{2}$ as great, and hence the ratio $\frac{24}{x}$ twice as great. As it is there are to be $\frac{24}{18}$ times as many days; whence it will make the ratio $\frac{24}{18}$ times as great as it was before, and we have

$$\tfrac{24}{18} \text{ times } \tfrac{12}{8} = \frac{24}{x}$$

when this consideration is involved.

3d. Again, we can in like manner show that the length being increased (from 95 to 380), $\tfrac{380}{95}$ *more men* will be required (*i. e.*, x will be made greater), and hence the ratio $\frac{24}{x}$ will be diminished—it will be only $\tfrac{95}{380}$ of what it would have been without this condition; and we have

$$\tfrac{95}{380} \text{ times } \tfrac{24}{18} \text{ times } \tfrac{12}{8} = \frac{24}{x}.$$

4th. In a similar manner we show that the width being diminished (from 12 to 9 *ft.*), only $\tfrac{9}{12}$ as many men would be required (*i. e.*, x would be made only $\tfrac{9}{12}$ as great). Hence the ratio $\frac{24}{x}$ is made $\tfrac{12}{8}$ as great, and we have

$$\tfrac{12}{8} \text{ times } \tfrac{95}{380} \text{ times } \tfrac{24}{18} \text{ times } \tfrac{12}{8} = \frac{24}{x}.$$

5th. In exactly the same manner we see that the depth being only $\frac{4}{6}$ as great, will require only $\frac{4}{6}$ as many men, or the ratio $\frac{24}{x}$ will be made $\frac{4}{6}$ as great; and we have

$$\tfrac{4}{6} \text{ times } \tfrac{95}{380} \text{ times } \tfrac{12}{8} \text{ times } \tfrac{24}{18} \text{ times } \tfrac{12}{8} = \frac{24}{x}, \text{ or}$$

$$\tfrac{4}{6} \times \tfrac{95}{380} \times \tfrac{12}{8} \times \tfrac{24}{18} \times \tfrac{12}{8} = \frac{24}{x}, \text{ or } 1 = \frac{24}{x}, \; x = 24.$$

The above having been given so much at length, we give simply the statement of the two remaining examples.

(7.)	Given case,	7 *mo.*,	6 *mi.*,	100 *men.*
	Required case,	5 *mo.*,	9 *mi.*,	x *men.*

We see from the problem that in 7 *mo.* 100 men have built 6 *mi.*; and the question now is, how many men will

it take to build the remaining 9 *mi.* in 5 *mo.* The proportion is

$$\tfrac{5}{7} \text{ of } \tfrac{8}{9} = \frac{100}{x}; \ \therefore x = 210,$$

and 110 additional men will be needed.

(8.) If 1 *yd.* shrink 13.99%, it yields 100% — 13.99%, or 86.01%; *i.e.*, .8601 *yd.* In like manner in the 2d case the yield by reason of shrinkage is .9133 *yd.* from 1 of cloth.

Given case, 8.14 *oz.* weight, .8601 *yd.* yield, 12.41 *oz.* raw wl.
Req'd " 7.43 " .9133 " x "

Proportion, $\frac{814}{743}$ of $\frac{8601}{9133} = \frac{12.41}{x}$; $\therefore x = 10.67$ *oz.*

PARTITIVE PROPORTION.

PAGE **135.**

(2.) $\frac{3+4}{3} = \frac{35}{x}$; $\therefore x = 15$. $\frac{3+4}{4} = \frac{35}{x'}$; $\therefore x' = 20$.

$\frac{2+5}{2} = \frac{35}{x}$; $\therefore x = 10$. $\frac{2+5}{5} = \frac{35}{x'}$; $\therefore x' = 25$.

$\frac{6+1}{1} = \frac{35}{x}$; $\therefore x = 5$. $\frac{6+1}{6} = \frac{35}{x'}$; $\therefore x' = 30$.

Or, $3x + 4x = 35$, $x = 5$; whence $3x = 15$, and $4x = 20$.
$2x + 5x = 35$, $x = 5$; whence $2x = 10$, and $3x = 15$.
$6x + x = 35$, $x = 5$; whence $6x = 30$.

(3.) $\frac{\frac{1}{2}+\frac{2}{3}+\frac{1}{6}}{\frac{1}{2}} = \frac{1}{x}$, $\frac{11}{15} = \frac{1}{x}$; $\therefore x = \frac{15}{11}$.

$\frac{\frac{1}{2}+\frac{2}{3}+\frac{1}{6}}{\frac{2}{3}} = \frac{1}{x'}$, $\frac{11}{10} = \frac{1}{x'}$; $\therefore x' = \frac{10}{11}$.

$\frac{\frac{1}{2}+\frac{2}{3}+\frac{1}{6}}{\frac{1}{6}} = \frac{1}{x''}$, $\frac{11}{6} = \frac{1}{x''}$; $\therefore x'' = \frac{6}{11}$.

Or, $\frac{1}{2}x + \frac{2}{3}x + \frac{1}{6}x = 1$, $x = \frac{10}{11}$; whence $\frac{1}{2}x = \frac{15}{11}$, $\frac{2}{3}x' = \frac{10}{11}$, and $\frac{1}{6}x = \frac{6}{11}$.

(4.) $\frac{3+5+8+2}{3} = \frac{7}{x}$, $\frac{18}{3} = \frac{7}{x}$; $\therefore x = 1\frac{1}{3}$.

$\frac{3+5+8+2}{5} = \frac{7}{x'}$, $\frac{18}{5} = \frac{7}{x'}$; $\therefore x' = 1\frac{7}{18}$.

$\frac{3+5+8+2}{8} = \frac{7}{x''}$, $\frac{18}{8} = \frac{7}{x''}$; $\therefore x'' = 3\frac{1}{8}$.

$\frac{3+5+8+2}{2} = \frac{7}{x'''}$, $\frac{18}{2} = \frac{7}{x'''}$; $\therefore x''' = \frac{7}{9}$.

Or, $3x + 5x + 8x + 2x = 7$, $x = \frac{7}{18}$; whence $3x = \frac{21}{18} = 1\frac{1}{6}$, $5x = \frac{35}{18} = 1\frac{17}{18}$, $8x = \frac{56}{18} = 3\frac{1}{9}$, $2x = \frac{14}{18} = \frac{7}{9}$.

(5.) $\frac{4+2+1}{4} = \frac{5000}{x}$, $\frac{7}{4} = \frac{5000}{x}$; $\therefore x = 2857\frac{1}{7}$.

$\frac{4+2+1}{2} = \frac{5000}{x'}$, $\frac{7}{2} = \frac{5000}{x'}$; $\therefore x' = 1428\frac{4}{7}$.

$\frac{4+2+1}{1} = \frac{5000}{x''}$, $\frac{7}{1} = \frac{5000}{x''}$; $\therefore x'' = 714\frac{2}{7}$.

Or, $4x + 2x + x = 5000$; whence $x = 714\frac{2}{7}$; and $2x = 1428\frac{4}{7}$; and $4x = 2857\frac{1}{7}$.

(6.) $\frac{800+700+1000+500}{800} = \frac{100}{x}$; $\therefore x = 26\frac{2}{3}$.

$\frac{800+700+1000+500}{700} = \frac{100}{x'}$; $\therefore x' = 23\frac{1}{3}$.

$\frac{800+700+1000+500}{1000} = \frac{100}{x''}$; $\therefore x'' = 33\frac{1}{3}$.

$\frac{800+700+1000+500}{500} = \frac{100}{x'''}$; $\therefore x''' = 16\frac{2}{3}$.

Or, $800x + 700x + 1000x + 500x = 100$, $x = \frac{1}{80}$; whence $800x = \frac{800}{80} = 26\frac{2}{3}$, $700x = \frac{700}{80} = 23\frac{1}{3}$, $1000x = \frac{1000}{80} = 33\frac{1}{3}$, $500x = \frac{500}{80} = 16\frac{2}{3}$.

ARITHMETICAL PROGRESSION.

PAGES 137, 138.

(*213.*) RULE.—To the first term add the product of the common difference multiplied by the number of terms less 1.

RULE.—Multiply half the sum of the extremes by the number of terms.

(1.) $l = a + (n-1)d = 5 + (8-1)3 = 26$; $s = \left(\frac{a+l}{2}\right)n = \left(\frac{5+26}{2}\right)8 = 124$.

(2.) $l = a + (n-1)d = 11 + (7-1)2 = 23$; $s = \left(\frac{a+l}{2}\right)n = \left(\frac{11+23}{2}\right)7 = 119$.

(3.) $l = 4 + (13-1)5 = 64$; $s = \left(\frac{4+64}{2}\right)13 = 442$.

(4.) The series being 1, 3, 5, 7, etc., $a = 1$, and $d = 2$. Hence $l = 1 + (50-1)2 = 99$, and $s = \left(\frac{1+99}{2}\right)50 = 2500$.

(5.) The series being 2, 4, 6, etc., $a = 2$, and $d = 2$. Hence, $l = 2 + (35-1)2 = 70$, and $s = \left(\frac{2+70}{2}\right)35 = 1260$.

(6.) The series being 7, 11, 15, etc., $a = 7$, and $d = 4$. Hence $l = 7 + (31-1)4 = 127$, and $s = \left(\frac{7+127}{2}\right)31 = 2077$.

(8.) $a = 132$, and $d = -7$. Hence $l = 132 + (13-1)(-7) = 132 - 84 = 48$, and $s = \left(\frac{132+48}{2}\right)13 = 1170$.

Query after (9). $s = \left(\frac{a+l}{2}\right)n$ cannot give a fraction when a, d, and n are integers, since as $l = a + (n-1)d$,

$s = \left(\frac{a + a + (n-1)d}{2}\right)n = \left(\frac{2a + (n-1)d}{2}\right)n = an + \frac{(n-1)n}{2}d$. Now the product an is integral, since both its factors are integers; and $\frac{(n-1)n}{2}$ is always integral if n is an integer, for if n is even, $\frac{n}{2}$ is an integer, but if n is odd, $n-1$ is even, and $\frac{n-1}{2}$ is an integer. Hence in any event $\frac{(n-1)n}{2}$ is an integer, and as by hypothesis d is an integer, $\left(\frac{(n-1)n}{2}\right)d$ is integral.

(10.) In the first $a = 5$, and $d = \frac{1}{3}$. Hence $l = 5 + (12-1)\frac{1}{3} = 8\frac{2}{3}$. $s = \left(\frac{5 + 8\frac{2}{3}}{2}\right)12 = 82$. In the second $a = 5$, and $d = -\frac{1}{3}$. Hence $l = 5 + (12-1)(-\frac{1}{3}) = 5 - \frac{11}{3} = 1\frac{1}{3}$. $s = \left(\frac{5 + 1\frac{1}{3}}{2}\right)12 = 38$.

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(*214.*) RULE.—Subtract the first term from the last, and divide the remainder by the number of terms less 1.

(*216.*) RULE.—Subtract the first term from the last, divide the remainder by the common difference, and add 1 to the quotient.

(*217.*) RULE.—Multiply the common difference by the number of terms less 1, and subtract the product from the last term.

(1.) $d = \frac{l-a}{n-1} = \frac{43-7}{13-1} = 3$.

(2.) $n = \frac{l-a}{d} + 1 = \frac{85-8}{7} + 1 = 12$.

(3.) $d = \frac{l-a}{n-1} = \frac{491-596}{21} = -5.$

(4.) $n = \frac{l-a}{d} + 1 = \frac{4-12}{-\frac{1}{8}} + 1 = \frac{-8}{-\frac{1}{8}} + 1 = 40 + 1 = 41.$

(*220.*) $s = \left(\frac{a+l}{2}\right)n$, $2s = an + nl$, $nl = 2s - an$, $l = \frac{2s}{n} - a.$

(*222.*) Produce the formula from $l = a + (n-1)d$, and $s = \left(\frac{a+l}{2}\right)n$. From the first $a = l - (n-1)d$, which substituted in the second, $s = \left(\frac{l-(n-1)d+l}{2}\right)n$ $= \left(\frac{2l-(n-1)d}{2}\right)n = ln - \frac{n(n-1)d}{2}.$

Pages 140, 141.

(1.) 1st. Given s, a, and l, to find d. Eliminating, $d = \frac{l^2 - a^2}{2s - a - l} = \frac{(78)^2 - 3^2}{2 \cdot 648 - 3 - 78} = 5.$

2d. Given s, a, and l, to find n. From $s = \left(\frac{a+l}{2}\right)n$, $n = \frac{2s}{a+l} = \frac{2 \cdot 648}{3 + 78} = \frac{1296}{81} = 16.$

(2.) 1st. Given l, n, and s, to find a. From $s = \left(\frac{a+l}{2}\right)n$, $\frac{2s}{n} = a + l$; $\therefore\ a = \frac{2s}{n} - l = \frac{2 \cdot 2100}{12} - 164 = 186.$

2d. $d = \frac{l-a}{n-1} = \frac{164-186}{12-1} = -2.$

(3.) We wish d. $d = \frac{l-a}{n-1}$. But n is not given.

Hence, finding it from $s = \left(\frac{a+l}{2}\right)n$, $n = \frac{2s}{a+l}$. Hence,

$$d = \frac{l-a}{\frac{2s}{a+l} - 1} = \frac{l^2 - a^2}{2s - a - l} = \frac{(17)^2 - 1^2}{2 \cdot 153 - 1 - 17} = 1.$$

Whence the series is 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17.

(4.) $a = 20$, $l = 59$, and $n = 14$. To find d, from $l = a + (n-1)d$, $d = \frac{l-a}{n-1} = \frac{59-20}{14-1} = 3$. Hence the series is 20, 23, 26, etc., to 59.

(5.) $a = 3$, $l = 4\frac{1}{2}$, and $n = 15$, to find d. $d = \frac{l-a}{n-1} = \frac{4\frac{1}{2} - 3}{15-1} = \frac{3}{28}$. Hence the series is 3, $3\frac{3}{28}$, $3\frac{6}{28}$, $3\frac{9}{28}$, $3\frac{12}{28}$, etc., to $4\frac{1}{28}$.

(6.) $a = 100$, $d = -5$, $n = 17$, to find l and s. $l = a + (n-1)d = 100 + (17-1)(-5) = 20$. $s = \left(\frac{100+20}{2}\right)17 = 1020$.

(7.) $d = \frac{l-a}{n-1} = \frac{7\frac{1}{2} - 4}{14} = \frac{1}{4}$.

(8.) $s = \left(\frac{1+24}{2}\right)24 = 300$.

(9.) $l = 120$, $n = 18$, $s = 1125$. $a = \frac{2s}{n} - l = \frac{2250}{18} - 120 = 5$.

GEOMETRICAL PROGRESSION.

PAGES 143, 144.

(227.) RULE.—From $1 = ar^{n-1}$. To find the last term of a Geometrical Progression when the first term,

ratio, and number of terms are given, multiply the first term by the power of the ratio whose exponent is 1 less than the number of terms.

From $s = \frac{lr - a}{r - 1}$, from the product of the last term by the ratio subtract the first term, and divide the remainder by the ratio minus 1.

Or, to find the sum directly from the first term and ratio, $s = \frac{a^r - a}{r - 1}$, *i. e.*, from that power of the first term whose exponent is the ratio, subtract the first term, and divide the remainder by the ratio minus 1.

(1.) $a = 5$, $r = 2$, $n = 8$. $l = ar^{n-1} = 5 \times 2^7 = 640$. $s = \frac{lr - a}{r - 1} = \frac{640 \times 2 - 5}{r - 1} = 1275$.

(3.) $a = 2$, $l = 1458$, $r = 3$. $s = \frac{lr - a}{r - 1} = \frac{1458 \times 3 - 2}{2} = 2186$.

(4.) $a = 7\frac{1}{4}$, $r = \frac{1}{2}$, $n = 6$. $l = ar^{n-1} = (7\frac{1}{4})(\frac{1}{2})^5 = \frac{29}{128}$.

Since this series is decreasing both numerator and denominator of $\frac{lr - a}{r - 1}$ are negative, and we may, if we choose, write $s = \frac{a - lr}{1 - r}$, since changing the signs of numerator and denominator of a fraction does not change its value, being equivalent to multiplying, or dividing, numerator and denominator by -1. Hence, in this case we have $s = \frac{7\frac{1}{4} - \frac{29}{128} \times \frac{1}{2}}{1 - \frac{1}{2}} = \frac{\frac{1856}{256} - \frac{29}{256}}{\frac{1}{2}} = \frac{1827}{128} = 14\frac{35}{128}$. But this change of signs in the formula is by no means necessary. Thus, without changing, we have in this case $s = \frac{lr - a}{r - 1} = \frac{\frac{29}{128} \times \frac{1}{2} - 7\frac{1}{4}}{\frac{1}{2} - 1} = \frac{\frac{29}{256} - \frac{1856}{256}}{-\frac{1}{2}} = \frac{-\frac{1827}{256}}{-\frac{1}{2}} = \frac{1827}{128} = 14\frac{35}{128}$.

(5.) $l = 1280$, $r = 2$, $n = 9$, $l = ar^{n-1}$, whence $a = \frac{l}{2^{n-1}} = \frac{1280}{2^8} = 5$.

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(1.) $s = \frac{a}{1-r} = \frac{1}{\frac{1}{2}} = 2$.

(2.) $s = \frac{a}{1-r} = \frac{\frac{1}{2}}{\frac{1}{4}} = 2$.

(3.) $s = \frac{a}{1-r} = \frac{14}{1-\frac{1}{7}} = \frac{14}{\frac{6}{7}} = 16\frac{1}{3}$.

(5.) $.\dot{3} = \frac{3}{10} + \frac{3}{100} + \frac{3}{1000}$ + etc. to infinity. Hence, $a = \frac{3}{10}$, $r = \frac{1}{10}$, and $s = \frac{\frac{3}{10}}{1-\frac{1}{10}} = \frac{\frac{3}{10}}{\frac{9}{10}} = \frac{3}{9}$.

$.0\dot{3} = \frac{3}{100} + \frac{3}{10000} + \frac{3}{1000000}$ + etc. to infinity. Hence, $a = \frac{3}{100}$, $r = \frac{1}{100}$, and $s = \frac{\frac{3}{100}}{1-\frac{1}{100}} = \frac{3}{99}$.

$.0\dot{3} = \frac{3}{100} + \frac{3}{1000} + \frac{3}{10000}$ + etc. Hence $s = \frac{\frac{3}{100}}{1-\frac{1}{10}} = \frac{3}{90} = \frac{1}{30}$.

$.0\dot{2}\dot{5} = \frac{25}{10000} + \frac{25}{1000000} - \frac{25}{100000000}$ + etc. $s = \frac{\frac{25}{10000}}{1-\frac{1}{100}} = \frac{25}{9900} = \frac{5}{1980}$.

$.02\dot{5} = \frac{25}{1000} + \frac{25}{100000}$ + etc. $s = \frac{\frac{25}{1000}}{1-\frac{1}{1000}} = \frac{25}{999}$.

PAGE 145.

(1.) Given a, r, s, to find l. Since these are all in $s = \frac{lr-a}{r-1}$, we have but to find the value of l in this formula. $(r-1)s = lr - a$, $lr = (r-1)s + a$, $l = \frac{(r-1)s+a}{r}$.

(2.) Given r, n, s, to find l. As these are not in either one of the two fundamental formulæ, we combine the

two, eliminating the a which is not involved in this problem. From $l = ar^{n-1}$, $a = \frac{l}{r^{n-1}}$. Substituting this in $s = \frac{lr - a}{r - 1}$, we have $s = \frac{lr - \frac{l}{r^{n-1}}}{r - 1} = \frac{l\left(r - \frac{1}{r^{n-1}}\right)}{r - 1}$

Whence $l = \frac{(r-1)s}{r - \frac{1}{r^{n-1}}} = \frac{(r-1)sr^{n-1}}{r^n - 1}$.

(3.) Since $l = ar^{n-1}$, $s = \frac{lr - a}{r - 1} = \frac{ar^n - a}{r - 1} = \frac{a(r^n - 1)}{r - 1}$.

(4.) From $l = ar^{n-1}$, $a = \frac{l}{r^{n-1}}$. Whence $s = \frac{lr - a}{r - 1}$

$= \frac{lr - \frac{l}{r^{n-1}}}{r - 1} = \frac{lr^n - l}{r^n - r^{n-1}}$

(5.) From $l = ar^{n-1}$, $a = \frac{l}{r^{n-1}}$.

6.) From $s = \frac{lr - a}{r - 1} = \frac{ar^n - a}{r - 1}$, $(r-1)s = a(r^n - 1)$, and $a = \frac{(r-1)s}{r^n - 1}$.

(7.) $s = \frac{lr - a}{r - 1}$, $s(r-1) = lr - a$, $a = lr - s(r-1)$.

(8.) $s = \frac{lr - a}{r - 1}$, $sr - s = lr - a$, $sr - lr = s - a$, $r = \frac{s - a}{s - l}$.

PAGE 146.

(1.) $s = \frac{lr - a}{r - 1} = \frac{1240029 \times 3 - 7}{3 - 1} = 1860040.$

(2.) $r = 3$, $n = 7$, $s = 1093$. $l = \frac{(r-1)sr^{n-1}}{r^n - 1} = \frac{(r^n - r^{n-1})s}{r^n - 1} = \frac{(3^7 - 3^6)1093}{3^7 - 1} = 729.$

(3.) $a = 1$, $l = 2048$, $s = 4095$. $r = \frac{s-a}{s-l} = \frac{4095 - 1}{4095 - 2048} = 2.$

(4.) $a = 1$, $r = 2$, $n = 12$. $s = \frac{a(r^n - 1)}{r - 1} = \frac{1(2^{12} - 1)}{2 - 1} = 4095.$

(5.) $s = \frac{1(2^{64} - 1)}{2 - 1} = 18{,}446{,}744{,}073{,}709{,}551{,}615$, being the number of grains of wheat.

One bushel = 7680 grains × 64 = 491,520 grains.

Hence, 18,446,744,073,709,551,615 ÷ 491,520 = 37,529,996,894,754, the number of bushels. This, at $1.50 per bushel, gives $56,294,995,342,131.

PERCENTAGE.

Page 150.

(3.) $A = b + br$, and since $p = br$, $A = b + p$; whence $p = A - b$.

(4.) From $p = br$, $b = \frac{p}{r}$; whence substituting in $A = b(1 + r)$, $A = \frac{p}{r}(1 + r)$, and $Ar = p(1 + r)$, $p = \frac{Ar}{1 + r} = \frac{A}{1 + r}r$.

(5.) From $A = b + br$, since $p = br$, $A = b + p$; whence $b = A - p$.

(6.) From $p = br$, $b = \frac{p}{r}$.

(7.) From $A = b + br$, $A = b(1 + r)$, and $b = \frac{A}{1 + r}$.

(8.) From $A = b + br$, $br = A - b$, and $r = \frac{A}{b} - 1$.

(9.) From $p = br$, $b = \frac{p}{r}$; and substituting in $A = b(1 + r)$, $A = \frac{p}{r}(1 + r) = \frac{p}{r} + p$, whence $r = \frac{p}{A - p}$.

(10.) From $p = br$, $r = \frac{p}{b}$.

(11.) From $A = b + br$, since $p = br$, $A = b + p$.

(12.) From $p = br$, $b = \frac{p}{r}$; whence substituting in $A = b + br = b(1 + r) = \frac{p}{r}(1 + r) = \frac{p(1 + r)}{r} = \frac{p}{r} + p$.

(*239.*) 1. To find the percentage from the base and rate, multiply the base by the rate.

2. To find the amount from the base and rate, multiply the base by 1 + the rate.

3. To find the percentage from the base and amount, subtract the base from the amount.

4. To find the percentage from the amount and rate, divide the product of the amount and rate by 1 + the rate.

5. To find the base from the amount and percentage, subtract the percentage from the amount.

6. To find the base from the rate and percentage, divide the percentage by the rate.

7. To find the base from the amount and rate, divide the amount by 1 + the rate.

8. To find the rate from the amount and base, divide the amount by the base, and diminish the quotient by 1.

9. To find the rate from the amount and percentage,

divide the percentage by the amount minus the percentage.

10. To find the rate from the percentage and base, divide the percentage by the base.

11. To find the amount from the base and percentage, take the sum of base and percentage.

12. To find the amount from the percentage and rate, multiply the percentage by 1 + the rate, and divide the product by the rate; or, add the percentage to the quotient of percentage divided by rate.

PAGES 152, 153.

(2.) BY THE FORMULÆ, $p = A - b = 270 - 250 = 20.$

$$r = \frac{A}{b} - 1 = \frac{270}{250} - 1 = \frac{27}{25} - \frac{25}{25} = \frac{2}{25} = .08.$$

ANALYSIS.—Since the amount is the sum of base and percentage, if percentage is subtracted from amount it leaves the base; hence in this case the percentage is 270 − 250, or 20. Since 250 of base gives 20 of percentage, 1 of base gives $\frac{20}{250}$, or .08. Hence the rate is .08, or the *rate per cent* is .08 of 100, or 8.

(3.) BY FORMULÆ, $b = \frac{p}{r} = \frac{20}{.08} = \frac{2000}{8} = 250.$

$$A = \frac{p(1+r)}{r} = \frac{20(1.08)}{.08} = \frac{2160}{8} = 270.$$

ANALYSIS.—Since every unit of base gives .08 percentage to give 20 as percentage requires as many units of base as .08 is contained times in 20, *i. e.*, $\frac{20}{.08} = 250$; and since amount is the sum of base and percentage, it is 250 + 20 = 270.

NOTE.—As the analysis of these cases is usually given in full in elementary treatises, and as most of the cases are fully given in the text-book to which this is a key, we shall here indicate the solutions only by the formulæ, which we have produced above.

(4.) $b = A - p = 126.44 - 10.44 = 116.$

$r = \frac{p}{A-p} = \frac{10.44}{126.44 - 10.44} = \frac{10.44}{116} = .09$, or 9%.

(5.) $b = \frac{A}{1+r} = \frac{126.44}{1.09} = 116.$ $p = \frac{Ar}{1+r} = \frac{126.44 \times .09}{1.09} = 10.44.$

(6.) $b = \frac{p}{r} = \frac{202.8}{.13} = 1560.$ $A = \frac{p}{r} + p = \frac{202.8}{.13} + 202.8 = 1560 + 202.8 = 1762.8.$

(7.) $p = \frac{Ar}{1+r} = \frac{1762.8 \times .13}{1.13} = 202.8.$ $b = \frac{A}{1+r} = \frac{1762.8}{1.13} = 1560.$

(8.) $p = \frac{Ar}{1+r} = \frac{157.17 \times .00\frac{3}{4}}{1.00\frac{3}{4}} = 1.17.$ $b = \frac{A}{1+r} = \frac{157.17}{1.00\frac{3}{4}} = 156.$

(9.) $A = b(1+r) = 358 \times 1.015 = 3633.7.$

(10.) $A = b(1+r) = 1342 \times 2.125 = 2851.75.$

(11.) $b = \frac{A}{1+r} = \frac{2851.75}{2.125} = 1342.$

(12.) $A = \frac{p(1+r)}{r} = \frac{600 \times 4}{3} = 800.$

PAGE 153.

(2.) $b = D + p = 328.7 + 17.3 = 346.$ $r = \frac{b-D}{b} = \frac{346 - 328.7}{346} = .05.$

(3.) $D = b - p = 346 - 17.3 = 328.7$. $r = \frac{p}{b} = \frac{17.3}{346} = .05$.

(4.) $D = \frac{p(1-r)}{r} = \frac{17.3(1-.05)}{.05} = \frac{17.3 \times .95}{.05} = 328.7$. $b = \frac{D}{1-r} = \frac{328.7}{1-.05} = \frac{328.7}{.95} = 346$.

(5.) $r = \frac{p}{D+p} = \frac{.5665}{14.8835 + .5665} = .03\frac{8}{15}$. $b = \frac{D}{1-r} = \frac{14.8835}{.96\frac{7}{15}} = 15.45$.

(6.) $b = \frac{D}{1-r} = \frac{14.8835}{.96\frac{7}{15}} = 15.45$. $p = br = 15.45 \times .03\frac{8}{15} = .5665$.

(7.) $r = \frac{b-D}{b} = \frac{15.45 - 14.8835}{15.45} = .03\frac{8}{15}$. $p = b - D = 15.45 - 14.8835 = .5665$.

(8.) $r = \frac{b-D}{b} = \frac{\frac{3}{5} - \frac{8}{15}}{\frac{3}{5}} = .2$, or 20%. $p = b - D = \frac{3}{5} - \frac{8}{15} = \frac{2}{15}$.

(9.) $D = b - br = \frac{3}{5} - \frac{3}{5} \times \frac{1}{5} = \frac{8}{15}$. $b = \frac{p}{r} = \frac{\frac{2}{15}}{.2} = \frac{2}{3}$.

(10.) $b = \frac{D}{1-r} = \frac{\frac{5}{12}}{1 - 1.50} = \frac{\frac{5}{12}}{-.50} = \frac{\frac{5}{12}}{-\frac{1}{2}} = -\frac{5}{6}$.

In an arithmetical sense this is absurd, since with any base a loss of 150%, or $1\frac{1}{2}$ times the base, is absurd. This appears in the solution by the base becoming negative. Again, $p = \frac{Dr}{1-r} = \frac{\frac{5}{12} \times \frac{3}{5}}{1 - \frac{3}{5}} = \frac{\frac{5}{8}}{-\frac{1}{2}} = -\frac{5}{4}$. Here the same arithmetical absurdity appears as in the first case.

(11.) 3. Since $br = p$, $D = b - br$ gives $D = b - p$, or $p = b - D$. 4. From $p = br$, $b = \frac{p}{r}$; whence $D = b - br = b(1 - r) = \frac{p}{r}(1 - r)$, and $p = \frac{Dr}{1 - r}$. 5. As in (3.) $p = b - D$; whence $b = D + p$. 7. From $D = b - br = b(1 - r)$. 8. From $D = b - br$, $br = b - D$, $r = 1 - \frac{D}{b}$. 9. From $p = br$, $b = \frac{p}{r}$; whence $D = b(1 - r)$ gives $D = \frac{p}{r}(1 - r)$, and $rD = p(1 - r) = p - pr$, $rD + pr = p$, $r = \frac{p}{D + p}$. 11. As in (3.) $p = b - D$; whence $D = b - p$. 12. $D = b(1 - r) = \frac{p}{r}(1 - r) = \frac{p(1 - r)}{r}$, or $\frac{p}{r} - p$, as before.

(13.) Since 100% is 100 of 100, or the whole of a number, $\frac{1}{2}$ of a number is $\frac{1}{2}$ of 100%, or 50%; $\frac{1}{3}$ of a number is $\frac{1}{3}$ of 100%, or $33\frac{1}{3}$%; $\frac{2}{5}$ of a number is $\frac{2}{5}$ of 100%, or 40%; $\frac{3}{4}$ of a number is $\frac{3}{4}$ of 100% of it, or 75%; $1\frac{1}{2} = \frac{3}{2}$ is $\frac{3}{2}$ of 100%, or 150%; 2 times a number is 2 times 100% of it, or 200% of it.

(14.) Since 20% of a number is .20 of it, it is $\frac{1}{5}$ of a number; 50% = .50 of a number, or $\frac{1}{2}$ of it; $33\frac{1}{3}\% = \frac{1}{3}$ of a number; 100% = the number itself (see above); 200% = 2 times a number; 75% = $\frac{3}{4}$ of a number; and 10% = $\frac{1}{10}$ of a number.

(15.) Since 400 is 100% of itself, 32 is $\frac{32}{400}$, or $\frac{8}{100}$ of 100% of it, *i. e.* 8%: since $\frac{1}{2}$ is 100% of itself, $\frac{1}{8}$ which is $\frac{1}{4}$ of $\frac{1}{2}$, is $\frac{1}{4}$ of 100%, or 80% of $\frac{1}{2}$: since 648 is 100% of itself, 38.88 is $\frac{38.88}{648}$, or $\frac{6}{100}$ of 100%, *i. e.* 6% of it: since 11

is 100% of itself, and $\frac{3}{4}$ is $\frac{3}{11}$, or $\frac{3}{44}$ of 11, it is $\frac{3}{44}$ of 100% of 11, or $6\frac{8}{11}\%$: since 12 is 100% of itself, and $1\frac{1}{3}$ is $\frac{1}{9}$ of 12, $1\frac{1}{3}$ is $\frac{1}{9}$ of 100%, or $12\frac{1}{9}\%$ of twelve.

PROFIT AND LOSS.

PAGES 156–159.

(1.) *Ans.* \$3.50 + 20% of \$3.50, or \$3.50 + $\frac{1}{5}$ of \$3.50 = \$3.50 + \$0.70 = \$4.20. Or, $A = b(1 + r)$ = \$3.50 × 1.20 = \$4.20.

(2.) $A = b(1+r)$. $10c. \times 1.15 = 11\frac{1}{2}c.$: $10c. \times 1.25 = 12\frac{1}{2}c.$: $10c. \times 1.10 = 11c.$

(8.) \$11.70 × 4 = \$46.80. $195 \times 4 \times 8\frac{3}{4}c.$ = \$67.08. \$67.08 − \$46.80 = \$20.28.

(9.) $2 = \frac{b - D}{b} = \frac{150 - 125}{150} = \frac{1}{6}$, or $16\frac{2}{3}\%$.

(10.) $b = \frac{p}{r} = \frac{3.33\frac{1}{3}}{.12\frac{1}{2}} = 26.66\frac{2}{3}$.

(11.) $p = br$. $12\frac{1}{2}c. \times .15 = 1\frac{7}{8}c.$: $\$6\frac{2}{3} \times .15 = \$1$: $38\frac{1}{2}c. \times .15 = 5\frac{7}{8}c.$: $\$33\frac{1}{3} \times .15 = \$5$: $\$18\frac{4}{9} \times .15$ = \$2.82.

(12.) \$15600 × 1.33⅓, or \$15600 × $1\frac{1}{3}$ = \$20800. Merchant sold for $\frac{3}{4}$ of \$20800 = \$1500. ∴ \$20800 − \$15600 = \$5200.

(14.) If I sell $\frac{3}{4}$ for what $\frac{1}{4}$ cost, I sell $\frac{1}{4}$ for $\frac{1}{3}$ cost, and $\frac{4}{4}$ for $\frac{4}{3}$ cost. Hence I lose $\frac{1}{3}$ cost, or $33\frac{1}{3}\%$. Or, $r = \frac{b - D}{b} = \frac{6 - 4}{6} = \frac{1}{3}$, or $33\frac{1}{3}\%$.

(15.) There is a loss of $\frac{1}{5}$ cost, or 20%.

(16.) $\frac{5}{4}$ is $1\frac{1}{4}$ times $\frac{4}{4}$; hence the selling price was $\frac{1}{4}$ more than cost, or 25%.

(17.) $5\% = \frac{1}{20}$: $10\% = \frac{1}{10}$: $20\% = \frac{1}{5}$: $25\% = \frac{1}{4}$: $50\% = \frac{1}{2}$: $100\% = 1$: $12\frac{1}{2}\% = \frac{1}{8}$.

(18.) \$1.09 × 2000 + \$1.12 × 1000 = \$3300, cost. \$3300 × $1.04\frac{6}{11}$ = \$3450, selling price. \$3450 ÷ 3000 = \$1.15.

(19.) 27*c.* is $1\frac{1}{2}$, or $\frac{2}{3}$ of cost. Hence cost is $22\frac{1}{2}c.$ $22\frac{1}{2}c. + \frac{1}{3}$ of $22\frac{1}{2}c. = 22\frac{1}{2}c. + 3\frac{3}{4} = 26\frac{1}{4}c.$: $22.5c. \times 1.3 = 29\frac{1}{4}c.$: $22.5c. \times 1.1 = 22.5c. + 2.25 = 24\frac{3}{4}c.$

(20.) \$4500 + \$500 = \$5000, cost. $\frac{\$800 - \$5000}{\$5000} = \frac{800}{5000}$ = .16, or 16%.

(21.) Losing $\frac{1}{5}$ he sold for $\frac{4}{5}$ cost. If \$164 is $\frac{4}{5}$, $\frac{1}{5}$ is \$41, and $\frac{5}{5}$ is \$205. Hence it cost $\frac{1}{40}$ of \$205, or \$5.12$\frac{1}{2}$ per meter.

(22.) $1\frac{1}{2}$ of \$7.00 = \$8.40 : \$8.40 ÷ 35 = 24*c.*

(23.) $D = b(1 - r) = \$15 \times .95 = \$14.25$. $\frac{\$14.25}{100} = 14\frac{1}{4}c.$

(24.) What cost 19.3*c.* was sold for 23.8*c.* Hence the gain was $23.8c. - 19.3c. = 4\frac{1}{2}c.$ A gain of $4\frac{1}{2}$ on 19.3 is a gain of $\frac{4.5}{19.3} = .23\frac{61}{193}$ on 1, or $23\frac{61}{193}\%$.

COMMISSION AND BROKERAGE.

PAGES 160, 161.

[We indicate briefly the operations.]

(1.) $\frac{\$375}{.05} = \$7500.$

(2.) \$3500 × $1.02\frac{1}{2}$ = \$3578.75.

(3.) $\frac{\$3700}{1.08} = \$3425.93-.$ Commission is reckoned on amount of money paid for land.

(4.) The collector has commission on all he collects, including his own pay. $D = b - br = b(1 - r) = \$75450 \times .95 = \$71677.50$.

(5.) $D = b(1 - r) = \$1562.50 + .95 = \$1484.375$.

(6.) $b = \frac{A}{1 + r} = \frac{\$8758.25}{1.02\frac{1}{2}} = \$8544.634$.

(7.) $b = \frac{A}{1 + r} = \frac{\$5650}{1.03\frac{1}{2}} = \$5467.74$.

$p = br = \$5467.74 \times .03\frac{1}{2} = \$182.26$.

(8.) $12 \times 20\,cwt. = 240\,cwt.$ $50c. \times 240 = \$120$. $\$120 + \$3 = \$123$, total charges. $240\,cwt. = 24000\,lbs.$ $25c. \times 24000 = \$6000$, the base. $r = .02\frac{1}{2}$. $p = br = \$6000 \times .02\frac{1}{2} = \$150$, commission. $\$6000 - \$150 - \$123 = \$5727$, sum remitted.

TAXES.

Pages 163–166.

(1.) $\$10000 + \$300 = \$10300$: $\$10300 - \$800 = \$9500$, to be raised. Tax on $\$1 = \frac{\$9500}{750000} = .01\frac{4}{15}$, or $1\frac{4}{15}\%$. $\$10000 \times .01\frac{4}{15} = \$126.66\frac{2}{3}$. $\$2200 \times .01\frac{4}{15} = \$27.86\frac{2}{3}$.

(2.) $\$750000 \times .02\frac{1}{2} = \$18750$, total tax. $\$18750 \times .95 = \$17812.50$, available. $\$18750 - \$17812.50 = \$937.50$, commission.

(3.) $\$3000 \div .95 = \$3157.8947$, collected. $\$3157.8947 \div .97 = \$3255.561$, levied. $\$3255.561 \div \$750000 = .00434074$, or $.43\frac{1}{2}\%$ nearly,—a little less than $\frac{1}{2}\%$.

(4.) 1st. $\$903434.50 \div 630000000 = 1.434023$ mills.

2d. $\frac{\$1.27}{\$5.736}$ of $\$5.736 = \$1.27$, tax for State Capitol.

$\frac{31500.00}{9034434.50}$ of \$5.736 = 20c., tax for University.
$\frac{28602.00}{9034434.50}$ of \$5.736 = 18c., tax for Agricultural College.
\$0.001434023 × 4000 = \$5.736, the man's State tax.

3d. \$0.001434023 × 1350 = \$1.9359, whole State tax.
$\frac{40000000}{9034434.50}$ of \$1.9359 = 43c., for State Capitol.
$\frac{33382.50}{9034434.50}$ of \$1.9359 = 7c., for Military purposes.
$\frac{28602 + 20000 + 31500}{9034434.50} = \frac{80102.00}{9034434.50}$. Hence,
$\frac{80102.00}{9034434.50}$ of \$1.9359 = 17c., for Educational purposes.
$\frac{28602.00}{9034434.50}$ of \$1.9359 = 6c., for Agricultural College.
$\frac{20000.00}{9034434.50}$ of \$1.9359 = 4c., for State Public School.
$\frac{31500.00}{9034434.50}$ of \$1.9359 = 7c., for University.

4th. \$31500 ÷ 630000000 = $\frac{1}{20}$ mill for University.
\$300000 ÷ 630000000 = .4762 mills for general purposes.

(6.) \$21963821.29 ÷ 1355401317 = 1.620466c. on \$1 valuation.

(7.)

Tax on		Tax on	
\$1 =	1.62046c.	\$10 =	16.2046c.
\$2 =	3.24093c.	\$20 =	32.4093c.
\$3 =	4.86139c.	\$30 =	48.6139c.
\$4 =	6.48186c.	\$40 =	64.8186c.
\$5 =	8.10233c.	\$50 =	81.0233c.
\$6 =	9.72279c.	\$60 =	97.2279c.
\$7 =	11.34326c.	\$70 =	\$1.134326
\$8 =	12.96372c.	\$80 =	\$1.296372
\$9 =	14.58419c.	\$90 =	\$1.458419.

The amounts on \$100 to \$900, and \$1000 to \$9000 are readily written from these.

(8.) Total valuation \$5680 + \$1050 = 6730.

From the Table, we have, Tax on \$6000, \$97.227
" " 700, 11.343
" " 30, .486
Tax on \$6730, \$109.06

Total valuation $548 + $85 = $633.

From the Table, we have, Tax on	$600,	$9.7227
" "	30,	.4861
" "	3,	.0486
Tax on	$633,	$10.26

Total valuation $8572 + $12765 = $21337.

Tax on	$20000,	$324.093
" "	1000,	16.2046
" "	300,	4.8613
" "	30,	.4861
" "	7,	.1134
Tax on	$21337,	$345.76

(9.) He was assessed on $125 ÷ .0162046, or $7713.83. For State purposes, $\frac{5983609.50}{21065831.25}$ of $125, or $28.59. For School purposes, $\frac{2225187.91}{21065831.25}$ of $125, or $5.69. Value of his property, $7713.83 ÷ $\frac{2}{3}$ = $11570.74.

(10.) He was assessed on $\frac{1}{2}$ of $20000, or $10000. His total tax was, therefore, $0.01620466 × 10000 = $162.0466; of which $\frac{2225187.91}{21065831.25}$, or $7.37, was School tax.

(11.) 1st. Tax on $1, $656491.61 ÷ 421285359, or $0.0015583. 2d. Tax on $5000 valuation, $0.0015583 × 5000 = $7.79. For Hospitals, $\frac{5778228}{656491.61}$ of $7.79, or 69 cents.

DUTIES.

Pages 167–169.

(2.) Value of Broadcloth, 13*s.* × 650 = 8450*s.*
" " Carpet, 3*s.* × 1950 = 5850*s.*
" " Drugget, 2$\frac{1}{3}$*s.* × 2560 = 5973$\frac{1}{3}$*s.*
Total, 20273$\frac{1}{3}$*s.*

30% on 20273⅓*s.*, is 6082*s.*

Value Lace, 2*s.* × 1246 = 2492*s.*
" Coach Lace, 1⅓*s.* × 1200 = 1100*s.*
Total value of Laces = 3592*s.*

25% on 3592*s.* is 898*s.*

Hence the total duty is 6082*s.* + 898*s.* = £349 = $4.8665 × 349 = $1698.41 in gold, or $1689.41 × 1.10 = $1868.25 in currency.

(3.) Net weight of a chest 67 *lbs.* — 9 *lbs.* = 58 *lbs.* Total cost 110 × 58 × 90*c.* = $5742. 40% on $5742 = $2296.80, the duty.

(4.) 50 × 63 *gal.* = 3150 *gal.* Less 3% is 3150 *gal.* × .97 = 3055½. 3055½ × 20*c.* = $611.10, the duty.

(5.) 54 *gal.* × 15 = 810 *gal.*, the amt. of wine. 810 *gal.* × .98 = 793.8 *gal.* after deducting leakage.

Specific duty, 20 *c.* × 793.8 = $158.76
Ad valorem duty, ¼ of 793.8 × 45*c.* = 89.30
Total duty, $248.06

(6.) $D = b(1 - r) = 80(1 - .05) = 80 \times .95 = 76$, the number of dozens subject to duty. $5.25 × 76 = $399, the cost.

Specific duty, 76 × 2½ × $3 = $627.00
Ad valorem duty, ½ of $399 = 199.50
Total duty, $826.50

(7.) 1160 *lbs.* × .9 = 1044 *lbs.*, net per *hhd.*, and 1044 *lbs.* × 20 = 20880 *lbs.* total net.

Specific duty, 20880 × 3*c.* = $626.40

(8.) £34.225 × .35 = £11.97875
£51.1875 × .40 = £20.475
£32.45375, Duty.

$4.8665 × 32.45375 = $157.936, in gold.
$157.936 × 1.12½ = $177.68, in currency.

(9.) .88*fr.* × 7563.8 = 6656.144*fr.*, cost. 40%, or ⅔ of 6656.144*fr.* = 2662.4576*fr.*, the *ad valorem* duty.

7583.8 *m.* = $\frac{22 \cdot 37}{56}$ of 7533.8, or 8271.85 *yd.*

Specific duty, 8271.85 × 8*c.* = $661.748
Ad valorem duty, 19.3*c.* × 6656.144 = 513.854
Total duty in gold, $1175.602
Duty in currency, $1175.602 × 1.11 = $1304.92.

(10.) At 112 *lb.* per *cwt.*, 5 *T.* 16 *cwt.* 3 *qrs.* 20 *lb.* = 13096. This, at $¼ per *lb.*, makes the invoice ¼ of 13096, or $3274. 20%, or ⅕ of $3274 = $654.80, the duty.

SIMPLE INTEREST.

PAGES 173–176.

(*262.*) RULE.—Divide the given interest by the product of the principal and rate; and the quotient is the time in years (or such period as the rate is named for).

(*263.*) RULE.—Subtract the principal from the amount and divide the remainder by the product of the principal and time.

DEM.—The amount minus the principal gives the interest, by definition. The principal multiplied by the time gives the interest at 100%, *i. e.* at rate 1. Hence the required rate is the same part of 1 that the given interest is of the interest at 100%. Therefore $\frac{\textit{given interest}}{\textit{int. at } 100\%} = \textit{required}$ rate.

(8.) 1 *yr.* 9 *mo.* 6 *da.* = 1.7½ *yr.* $i = prt$ = $240 × $\frac{7}{100}$ × $\frac{5 \cdot 3}{8}$ = $29.68.

Or, 7% being .07 of the principal, the interest for 1 *yr.*

is \$240 $\times .07 =$ \$16.80, and for $1.7\frac{1}{2}$ *yr.*, \$16.80 $\times 1.7\frac{1}{2}$, or \$29.68.

NOTE.—When specific dates are not given, 360 *da.* are called a year.

As the elementary analysis is made familiar in the text, we shall here give only the outlines of solutions by the formulæ.

(9.) 2 *yr.* 7 *mo.* 27 *da.* $= \frac{319}{10 \times 12}$.

$$\therefore i = prt = \$\cancel{584} \times \tfrac{1}{\cancel{100}} \times \frac{319}{10 \times \cancel{12}_{2}} = \$.292 \times 319 = \$93.15.$$

(10.) $p = \frac{i}{rt} = \frac{\$1.2924}{.06 \times 1\frac{1}{2}} = \frac{\$1.2924}{.09} = \$14.36.$

(11.) $p = \frac{i}{rt} = \frac{\$76.095}{.06 \times \frac{19}{24}} = \frac{\$7609.5}{\frac{19}{4}} = \frac{\$30438}{89} = \$342.$

(12.) $p = \frac{i}{rt} = \frac{\$295.887}{.10 \times 8.7} = \frac{\$295.887}{.87} = \$340.10.$

(13.) $t = \frac{i}{pr} = \frac{22.08}{120 \times .08} = \frac{22.08}{9.6} = 2.3.$

(14.) $t = \frac{A-p}{pr} = \frac{178-150}{150 \times .1} = \frac{28}{15} = 1.86\frac{2}{3}$. $1.86\frac{2}{3}$ *yr.* $= 1$ *yr.* 10 *mo.* 12 *da.*

(15.) $t = \frac{A-p}{pr} = \frac{441.84-420}{420 \times .08} = \frac{21.84}{33.6} = .65$. .65 *yr.* $= 7$ *mo.* 24 *da.*

(16.) 2 *yr.* 9 *mo.* 12 *da.* $= 2.7\frac{1}{2}$ *yr.*

$r = \frac{i}{pt} = \frac{13.26}{40 \times 2.7\frac{1}{2}} = \frac{39.78}{334} = .11\frac{\cancel{44}}{\cancel{111}}$, or $11\frac{\cancel{44}}{\cancel{111}}\%$.

(17.) $r = \frac{A-p}{pt} = \frac{157.375-125}{125 \times 3\frac{1}{2}} = \frac{32.375}{4.375} = .07\frac{2}{7}$, or $7\frac{2}{7}\%$.

(18.) $r = \frac{A-p}{pt} = \frac{46.712-36}{36 \times 3.7\frac{7}{50}} = \frac{10.712}{133.9} = .08$, or 8%.

(19. 5 *yr.* 5 *mo.* 15 *da.* $= \frac{431}{144}$ *yr.*

$A = p(1 + rt) = \$45(1 + \frac{2}{25} \times \frac{431}{144}) = \frac{\$45 \times 431}{300} =$ $\$15 \times 4.31 = \$64.65$.

(20.) $A = p(1 + rt) = \$746.25(1 + .05 \times 1.8\frac{1}{2}) =$ $\$746.25 \times 1.09\frac{1}{2} = \$815.90$.

(21.) $r = \frac{i}{pt} = \frac{224.33\frac{1}{3}}{750 \times 3\frac{111}{180}} = .08$, or 8%.

(22.) $t = \frac{i}{pr} = \frac{224.33\frac{1}{3}}{750 \times .08} = 3\frac{111}{180}$. $3\frac{111}{180}$ *yr.* $= 3$ *yr.* 8 *mo.* 26 *da.*

(23.) $p = \frac{i}{rt} = \frac{\$224.33\frac{1}{3}}{.08 \times 3\frac{111}{180}} = \frac{\$224.33\frac{1}{3}}{.2991} = \$750$.

(25.)

1872	6	10	30	24
1870	1	15	12	4.833+
2	4	25		2.4⅓

$\$250 \times .06 =$ 15

$\$36.04$

(26.)

1873	5	12	30	22
1871	2	20	12	2.733+
2	2	22		2.227+

$\$192.35 \times .05 =$ $\$ 9.61\frac{1}{4}$

$\$21.41$

(27.)

1875	12	1	30	26
1872	9	5	12	2.866+
3	2	26		3.2388+

$\$370.125 \times .07 =$ $\$25.908$

$\$83.92$

(28.)

1877	1	16	30	18
1874	10	28	12	2.6
2	2	18		2.2⅛

$\$500.1875 \times .05 =$ $\$40.015$

$\$88.70$

(29.) Time $2.71\frac{1}{18}$ *yr.* $\$734.625 \times .10 = \$73.4625$. $\$73.4625 \times 2.71\frac{1}{18} = \$205.29$.

EXACT INTEREST.

PAGE 177.

(2.) Time $4\ yr. + (11 + 31 + 30 + 31 + 30 + 31 + 31 + 30 + 23)\ da. = 4\ yr. + 248\ da. = 4\frac{248}{360}\ yr.$ $\$58.40 \times .08 \times 4\frac{248}{360} = \$21.86.$

(3.) Time $(7 + 31 + 30 + 31 + 30 + 10)\ da. = 139\ da.$ $\$500 \times (1 + .07) \times \frac{139}{360} = \$521.67.$

COMPOUND INTEREST.

PAGES 178–181.

(2.) If $P = 1$, $A = P(1 + r)^n = (1 + r)^n$; hence the amount of \$1 for time n, at rate r, is $(1 + r)^n$.

(3.) The formula for the last term of a geometric series is $l = ar^{n-1}$. Now if $a = P$, $r = 1 + r$, and $n = n + 1$, we have $l = P(1 + r)^n$. Or, we may write the amounts of P dollars for several years thus:

At date.	*End of 1st yr.*	*2d.*	*3d.*	*4th.*	*5th.*
P	$P(1+r)$	$P(1+r)^2$	$P(1+r)^3$	$P(1+r)^4$	$P(1+r)^4$
1st term.	*2d.*	*3d.*	*4th.*	*5th.*	*6th.*

Hence for the nth year it is $P(1+r)^n$; n is 1 less than the number of terms in the series since the series starts with P.

(4.) From the table the amount of \$1, at 7% for 8 *yr.*, is \$1.718186. Hence for \$300 it is $\$1.718186 \times 300 = \$515.46$. At 10%, $\$2.143589 \times 300 = \$643.08$.

(5.) $\$1.304773 \times 550 = \$717.63$.

$\$717.63 - \$550 = \$167.63$.

$\$1.229874 \times 550 = \$676.43$. $\$676.43 - \$550 = \$126.43$.

$\$1.092727 \times 550 = \$601$. $\$601 - \$550 = \$51$.

Since $(1 + r)^{15} = (1 + r)^{10} \times (1 + r)^5$, $\$1.343916 \times 1.159274 \times 550 = \$856.88$. $\$856.88 - \$550 = \$306.88$.

Since $(1 + r)^{24} = (1 + r)^8 \times (1 + r)^8 \times (1 + r)^8$,

$(1.26677)^3 = 2.03279$, amt. of \$1 for 25 *yr.* \$2.03279 × 550 = \$1118.03. \$1118.03 − \$550 = \$568.03.

Since $(1 + r)^{18} = (1 + r)^{10} \times (1 + r)^8$, \$1.343916 × 1.26677 = \$1.702432. \$1.702432 × 550 = \$936.34. \$936.34 − \$550 = \$386.34.

$(1+r)^{28} = (1+r)^{10} \times (1+r)^{10} \times (1+r)^8$. $(1.343916)^2$ × 1.159274 = 2.093777. \$2.093777 × 550 = \$1151.58. \$1151.58 − \$550 = \$601.58.

$(1 + r)^{31} = (1 + r)^{10} \times (1 + r)^{10} \times (1 + r)$. $(1.343916)^3 \times (1.03) = 1.860294$. \$1.860294 × 550 = \$1023.15. \$1023.16 − \$550 = \$473.16.

(6.) \$1.126162 × 3500 = \$3941.57. At 2%.
\$1.265319 × 3500 = \$4428.62. At 4%.
\$1.229255 × 3500 = \$4302.39. At 3½%.
\$1.973822 × 3500 = \$6908.38. At 12%.

(7.) Since the interest for ½ a year at 8% is the same as for a whole year at 4%, we treat the case of interest compounded semi-annually, at 8% per annum, for 3 *yr.* as for 6 periods at 4%. Thus $(1.04)^6 = 1.265319$. Hence \$1.265319 × 500 = \$632.66, the amount of \$500 at 8%, per annum, compounded semi-annually for 3 *yr.*, and \$632.66 − \$500 = \$132.66, the interest.

(8.) $(1.035)^{10} = 1.410599$. \$1.410599 × 75 = \$105.79. \$105.79 − \$75 = \$30.79.

$(1.05)^{10} = 1.628895$. \$1.628895 × 75 = \$122.17. \$122.17 − \$75 = \$47.17.

(9.) $(1.015)^2 = 1.030225$. $(1.030225)^2 = (1.015)^4 = 1.061363$. $(1.061363)^2 = (1.015)^8 = 1.126492$. \$1.126492 × 2500 = \$2816.23.

$(1.02)^8 = 1.171659$. \$1.171659 × 2500 = \$2929.15.

$(1.0125)^8 = (1.0125)^4 \times (1.0125)^4 = \{[(1.0125)^2]^2\}^2 = 1.104482$. \$1.104482 × 2500 = \$2761.21.

(10.) Amt. of \$1 for 3 *yr.* at 7%, \$1.225043
" " " " 2 *yr.* " " 1.1449
2) .080143
Int. for the last $\frac{1}{2}$ *yr.* 040071

Int. of \$1 for $2\frac{1}{2}$ *yr.* at 7%, \$1.1449 + \$0.040071 = \$1.184971, and for \$450, 450 times \$1.184971 = \$533.24. \$533.24 — \$450 = \$83.24.

Int. for 4th *yr.* \$1.310796 — \$1.225043 = \$0.085753. 10 *mo.* = $\frac{5}{6}$ *yr.* $\frac{5}{6}$ of \$0.085753 = \$0.071461. Hence the amt. of \$1 for 3 *yr.* 10 *mo.* is \$1.225043 + \$0.085753 = \$1.296504. \$1.296504 × 450 = \$583.43. \$583.43 — \$450 = \$133.43.

(14.) From the table we find that, at 8%, \$1 amounts to \$2.158925 in 10 *yr.*, and \$1.999005 in 9 *yr.* Hence we see that the time required for it to double (become \$2) is between 9 *yr.* and 10 *yr.* (in fact but little more than 9 *yr.*) To find the fractional part of the 10th year, we have $\frac{2 - 1.999005}{2.158925 - 1.999005} = \frac{995}{159920}$. $\frac{995}{159920}$ of 365 *da.* is 2 *da.*+. Hence the time is 9 *yr.* 2 *da.*+.

As above we have $\frac{2.000000 - 1.992563}{2.171893 - 1.992563} = \frac{7437}{179330}$. $\frac{7437}{179330}$ of 365 *da.* = 15 *da.* Hence the time is 8 *yr.* 15 *da.*

$\frac{2.000000 - 1.948717}{2.143589 - 1.948717}$ of 365 *da.* = $\frac{51283}{194872}$ of 365 *da.* = 96 *da.* Hence the time is 7 *yr.* 96 *da.*

$\frac{2.000000 - 1.973822}{2.210681 - 1.973822}$ of 365 *da.* = 40 *da.* ∴ 6 *yr.* 40 *da.*

(15.) If \$500 at 6% amounts to \$669.112 in a given time, \$1 will amount to $\frac{1}{500}$ of \$669.112 = \$1.338224. And from the table we find that \$1.338224 is the amt. of \$1 for 5 *yr.* at 6%, very nearly.

(16.) $\frac{\$150 + \$75}{150} = \$1.5$. $\frac{1.500000 - 1.477455}{1.551328 - 1.477455}$ of 365 *da.* = 111 *da.* Hence the time is 8 *yr.* 111 *da.*

(17.) The rate is .02 per quarter. $\frac{\$240 + \$46.82}{240} =$ \$1.195083. From the table we find that this is the *amt.* of \$1 for 9 intervals, or, in this case, for $\frac{1}{4}$ *yr.* = $2\frac{1}{4}$ *yr.*

(19.) $A = P(1 + r)^n$. $(1 + r)^5 = \frac{A}{P} = \frac{304.16}{250} =$ 1.21664. $1 + r = \sqrt[5]{1.21664} = 1.04$, very nearly; and $r = .04$, or 4% semi-annually, *i. e.* 8% annually.

ANNUAL INTEREST.

PAGES 182, 183.

(1.) The first payment of interest having been made, the computation is for 4 *yr.* The simple interest accruing on the note for each year being \$60, we have the following:

Amt. of \$60 for	$3\frac{3}{365}$ *yr.* =	\$ 78.049
" " \$60 "	$2\frac{3}{365}$ *yr.* =	72.049
" " \$60 "	$1\frac{3}{365}$ *yr.* =	66.049
Amt. of face of note \$600 for	$1\frac{3}{365}$ *yr.* =	660.493
	Total amt. of note,	\$876.64

NOTE 1.—Observe that in cases where no payments have been made on a note bearing Annual Interest, the *amounts of the several deferred payments of Annual Interest constitute a decreasing arithmetical series*, which, if grace is reckoned, has for the first term the amount of 1 *yr.* int. on the face of the note for the number of years less 1, + the 3 *da.* grace; for the common difference 1 year's interest on the annual interest; and for the number of terms the number of years − 1. To the sum of this series add the amount of the note for $1\frac{3}{365}$ *yr.* and the sum is the total amount of the note.

When grace is not reckoned on such a note the series becomes one in which the first term is the interest on the face of the note

for 1 *yr.* (the last year); the common difference is the interest on this interest for 1 *yr.*; and the number of terms is the number of years. To the sum of this series add the principal, and the sum is the amount of the note.

The solution of the above, both with and without grace, by this method is as follows:

With grace, $a = 78.049$, $d = 6$, $n = 3$, to find S. $S = \frac{a+2}{2}n$, and $l = a - r(n-1)$, whence $s = \frac{2a + d(n-1)}{2}n = \frac{156.098 - 6 \times 2}{2} \times 3 = 21.6147$. To this add the amount of \$600 for $1\frac{1}{365}$ *yr.*, *i. e.* \$660.493, and we have the amount of the note, \$876,64. as above.

Without grace, we have $a = 60$, $d = 6$, and $n = 4$. Hence $s = \frac{2a + d(n-1)}{2}n = \left(\frac{2 \times 600 + 6(4-1)}{2}\right)4 = 276$, which is the interest due on the note. Hence the amount is \$876.

NOTE 2.—There may be a question as to how grace should be reckoned on such a note; but, in the absence of any established custom, the author suggests that here presented, which makes each payment of annual interest due at the exact dates, but allows grace at the final settlement. To allow grace on the payments, it should also be included in computing the annual interest. Thus, in the above case, the interest falling due annually would be \$60.493, instead of \$60. But this would reckon interest twice for the days of grace. Or, otherwise, we might call the first interest due 3 *da.* after the expiration of the first year, and reckon simple interest on this to the time of settlement (including the final grace). This, in the above case, would be \$60.493 on interest for 3 *yr.* Then the other payments would be \$60 *each*, and on interest for exact years. This gives in the above example

\$600 + \$60.493 + \$18.1479 + \$72 + \$66 + \$60 = \$876.64,

the same as the method above.

(2.) Int. for 1 *yr.* at 7%, on \$175.50 = \$12.285.

Amt. of	\$12.285	for	$3\frac{3}{365}$	*yr.*,	\$14.8718
"	"	"	$2\frac{3}{365}$	*yr.*,	14.0119
"	"	"	$1\frac{3}{365}$	*yr.*,	13.1519
"	\$175.50	"	$1\frac{3}{365}$	*yr.*,	187.885
			Whole amount,		\$229.92

(3.) The date of maturity being given no grace is reckoned, and as months are included without specifying what month, exact time can not be reckoned. Hence 360 days is called a year.

Int. on \$500 for 1 *yr.* at 8% is \$40.

Amt. of \$40 for	$2\frac{53}{72}$ *yr.*,	\$48.755
" " " "	$1\frac{53}{72}$ *yr.*,	45.555
" " " "	$\frac{53}{72}$ *yr.*,	42.355
" "\$500 "	$\frac{53}{72}$ *yr.*,	529.444
	Whole amt.,	\$666.11

(4.) The time for which interest is unpaid is $1\frac{13}{18}$ *yr.*

Int. on \$350 for 1 *yr.*, at 5%, is \$17.50.

Amt. of \$17.50 for	$\frac{13}{18}$ *yr.*,	\$18.13
" " \$350 "	$\frac{13}{18}$ *yr.*,	362.64
	Whole amt.,	\$380.77

(5.) The amounts are \$13.80 + \$13.08 + \$12.36 + \$206.00 = \$245.24.

(6.) Reckoned as deferred annual interest, without grace, we have \$70 + \$65 + \$60 + \$55 + \$550 = \$800. At compound interest we have the amount \$1.61051 × 500 =\$805.255. Diff. = \$5.255.

PARTIAL PAYMENTS.

PAGES **186, 187.**

(3.) 360 *da.* to year.

Principals.	*Time.*	*Interest.*	*Payments.*
\$840	$1\frac{1}{5}$ *yr.*	\$70.56	\$144.56
766	$1\frac{7}{18}$ *yr.*	78.64	495.75
348.89	$\frac{77}{120}$ *yr.*	15.668	364.56 Due.
	365 *da.* to year.		
\$840	$1\frac{73}{365}$ *yr.*	\$70.56	144.56
766	$1\frac{161}{365}$ *yr.*	78.446	495.75
348.696	$\frac{335}{365}$ *yr.*	15.716	364.41 Due.

(4.) 360 *da.* to year.

Principals.	*Time.*	*Interest.*	*Payments.*
$650	$\frac{22}{60}$ *yr.*	$18.864	$242.36
427	$1\frac{3}{60}$ *yr.*	29.263	144.90
311.563	$1\frac{1}{60}$ *yr.*	25.547	267.30
69.81	$1\frac{7}{60}$ *yr.*	4.676	74.49 Due.

365 *da.* to year.

$650	178 *da.*	$19.03	$242.36
427.17	1 *yr.* 53 *da.*	29.35	144.90
311.62	1 *yr.* 134 *da.*	25.56	267.30
69.88	1 *yr.* 44 *da.*	4.697	74.58 Due.

(5.) As no payment exceeds the interest due at the time it was made, the sum of the payments is taken from the amount of the note for the whole time, 5 *yr.* 7 *mo.* 19 *da.*; or, exact time, 5 *yr.* 231 *da.*

Amt. of $4000 for 5 *yr.* 7 *mo.* 19 *da.*, 10%, is 6254.44\frac{4}{9}$ (common method); this less $1000 gives 5254.44\frac{4}{9}$ due on the note.

Amt. of $4000 for 5 *yr.* 231 *da.* (exact time), at 10%, is $6253.15; this less $1000 gives $5253.15 due on the note.

MERCHANT'S RULE.

PAGES **187, 188.**

(1.) Amt. of $460 for 1 *yr.* at 10%, ... $506.00

Payments.	*Time.*	*Amounts.*	
$120	325 *da.*	$130.68	
50	240 *da.*	53.29	
100	170 *da.*	104.66	
80	85 *da.*	81.86.	Total pay'ts, $370.49
		Balc. due, ...	$135.51

(2.) Amt. of \$350 for 222 *da.*, at 7%, ... \$364.90

Payments.	*Time.*	*Amounts.*	
\$60	157 *da.*	\$61.81	
80	92 *da.*	81.41	
100	50 *da.*	100.96.	Total pay'ts, \$244.18
		Balc. due, ...	\$120.72

CONNECTICUT RULE.

PAGES 189, 190.

Amt. due Aug. 1, 1871, as per solution in book, \$686.1346
Amt. of \$35 for 8½ *mo.*, ... 36.4875
Due after pay't Aug. 1, 1871, ... \$649.6471
Int. for 1 *yr.*, ... 38.9788
Amt. Aug. 1, 1872, ... \$688.6259
Amt. of \$260 for 6 *mo.*, ... 267.80
Due Aug. 1, 1872, ... \$420.83

(3.) The common method of computing time is usually used with the Connecticut rule (360 *da.* to a year).

BY U. S. COURT RULE.

Amt. of \$1000 for 3 *mo.* 18 *da.*, . . . \$1021
Payment, ... 500
Balc. July 28, 1867, ... \$521
Amt. of \$521 for 9 *mo.* 9 *da.*, ... \$549.264
Sum of three pay'ts (8 + 12 + 125), . 145.
Balc. May 7, 1868, ... \$404.264
Int. on this balc. for 4 *mo.* 26 *da.*, . . 11.47
Amt. Oct. 3, 1868, ... \$415.734
Payment, ... 200.
Balc. due Oct. 3, 1868, ... \$215.734
Int. on this for 3 *mo.* 12 *da.*, ... 6.793
Amt. Mar. 15, 1869, ... \$222.527
Payment, ... 50.
Balc. due Mar. 15, 1869, ... \$172.527
Int. on this for 2 *mo.* 18 *da.*, ... 2.608
Due June 3, 1869, ... \$175.135

BY THE CONNECTICUT RULE.

Amt. of $1000 for 1 *yr.*,		$1070.00
Amt. of $500 for 8 *mo.* 12 *da.*,	$524.50	
" " 8 " 3 *mo.* 27 *da.*,	8.182	
" " 12 " 1 *mo.* 15 *da.*,	12.105	544.787
Balc. due Apr. 10, 1868, after pay'ts,		$525.213
Int. on this balc. for 1 *yr.*,		36.7649
Amt. of note, Apr. 10, 1869,		$561.979
Amt. of $125 for 11 *mo.* 3 *da.*,	$133.094	
" " 200 " 6 *mo.* 7 *da.*,	207.2722	
" " 50 " 25 *da.*,	50.243	$390.61
Amt. April 10, 1869,		$171.37
Int. on this balc. 1 *mo.* 23 *da.*,		1.77
Due June 3, 1869,		173.14

PARTIAL PAYMENTS ON ANNUAL INTEREST NOTES.

PAGE **193.**

(2.) Int. on $1000, 1 *yr.*, at 8%, $80
Amt. of payt. ($50) for 207 *da.*, 52.27
Deferred int., $27.73

Amt. of $27.73 for 2 *yr.*,		$32.167
" " $80 " 1 *yr.*,		86.40
" " $1000 " 1 *yr.*,		1080.00
Amt. of note, July 27, 1876,		$1198.567
Amt. $150 payt. for 311 *da.*,	$160.225	
Payment July 27, 1876,	200	360.225
Balc. due July 27, 1876,		$838.342
Int. on this for 1 *yr.*,		67.067
Amt. July 27, 1877,		$905.41
Amt. of $250 payt. for 174 *da.*,		259.53
Balc. due July 27, 1877,		$645.88
Int. on this for 36 *da.*,		5.096
Amt. due Sept. 1, 1877,		$650.98

(3.) Int. on \$500, 1 *yr.*, 6%, \$30, deferred 1 *yr.*, since the \$15 paid Feb. 10, 1874, was less than the interest then due.

Amt. of \$30 for 1 *yr.*,	\$ 31.80	
Amt. of \$500 for 1 *yr.*,	530.00	\$561.80

Inasmuch as the sum of the two payments, \$15 Feb. 10, 1874, and \$25 Aug. 15, 1874, exceeds the interest due at the latter date, this \$40 is on interest the remainder of that year, 296 *da.*, and the \$150 paid May 17, 1875, is also to be on interest the remainder of the year, 21 *da.* These amounts are \$41.946, and 150.518, or in all \$192.464. Hence we have,

Balc. due June 7, 1875, \$561.80 — \$192.464 =	\$369.336
Int. on this balc. for 1 *yr.*, ...	22.16
Amt. due June 7, 1876, ...	\$391.496
The \$20 paid Jan. 13, 1876, being more than the interest then due, we take the amt. for 145 *da.*, to the next rest, ...	20.477
Balc. due June 7, 1876, ...	\$371.02
Interest on this for 133 *da.*, ...	8.11
	\$379.13

When payts. are made "on interest accruing."

Amt. of \$500, 1 *yr.*, \$530. As by the stipulation the \$15 paid this year was "on int. accruing," applying it, there remains due June 7, 1874, \$515.

Amt. of \$515 for 1 *yr.*, \$545.90. During this year \$175 have been paid, paying up the "interest accruing," and leaving \$175 — \$45.90 = \$129.10, as payt. on the principal made May 17, 1875. The amt. of this at the next rest, June 7, 1875, 21 *da.*, is \$129.55. Applying this there remains \$500 — \$129.55 = \$370.45, due June 7, 1875.

Amt. of \$370.45 for 1 *yr.*, \$392.68. Deduct the \$20

paid during this year and $372.68 remains due June 7, 1876.

Amt. of $372.68 for 133 *da.*, *i. e.*, to Oct. 18, 1876, $380.83.

NOTE.—The legal provision in N. H. for endorsements "on interest accruing" is specially designed to cover cases in which payments are made of *less amount than the interest accruing*. Thus if I owe Mr. A $2000 on interest at 10%, and can pay him during the year $150 (at one or several payments), but cannot pay an amount equal to the interest accrued, according to the rule when the provision "on interest accruing" is not made, I should lie out of the use of my money till such time as I could pay up the entire interest accrued. Now suppose this should continue 5 *yr.* and I should thus pay $150 per *yr.*, I should lose the use of all the payments from the time they were made to the end of the 5 *yr.* Hence the debtor will seek to have his payments "on interest accruing" if he can not meet the entire accrued interest.

If, however, by oversight, or otherwise, the debtor has made a payment greater than the interest accrued, and has allowed it to be endorsed as "on interest accruing," only so much of it could be claimed "on interest accruing" as would cover the interest due at the close of that period, or year. The next year's interest is not yet accruing. Hence the balance of any such payment would be treated in the ordinary way provided for cases in which payments exceed the interest.

EXAMPLE.

$2000.

Three years after date, for value received, I promise to pay C. N. Jones, or bearer, *Two Thousand Dollars*, with annual interest at 10 per cent.

ANN ARBOR, May 7, 1876. W. W. BEMAN.

Endorsements: $150 on the 1st day of April each year.

What was the amount due on this note if the endorsements were "on interest accruing?" What if this phrase were not inserted?

If "on interest accruing," at May 7, 1874 there would be an unpaid balance of interest of $50. May 7, 1875 there would be $200 + $50 + $5 = $255 interest due. Deducting the $150 would leave

$105 interest unpaid. May 7, 1876 there would be $200 + $105 + $10.50 = $315.50 interest due. Deducting the payment $150, and adding the principal, we have $315.50 − $150 + $2000 = $2165.50, the amount due.

If the payments were not specified to be "on interest accruing," the amount due May 7, 1876 would be $200 + $40 + $200 + $20 + $200 + $2000 − $450 = $2210.

(*281.*)

Ex. 1.

Amt. of $500 for 1 *yr.*, ...		$525.
Amt. of $60 for 204 *da.*, $61.677		
Amt. of $80 for 89 *da.*,	80.975	142.65
Amt. due June 7, 1874, ...		$382.35
Amt. due June 7, 1875, ...		401.467
Amt. due June 7, 1876, ...		421.54
Amt. due June 7, 1877, ...		442.617
Amt. of $150 payt. for 298 *da.*, .		156.123
Balc. due June 7, 1877, ...		$286.494
Amt. for 132 *da.*, ...		291.674.

(*283.*)

RULE.—*To find one of the equal annual payments necessary to discharge both principal and interest of a note by the U. S. Court rule.*—Multiply the interest on the principal for 1 *yr.* by that power of 1 + the rate indicated by the number of payments, and divide this product by 1 less than the above power of 1 + the rate.

(2.) $x=\frac{br(1+r)^n}{(1+r)^n-1}=\frac{\$500\times\frac{7}{100}(1.07)^3}{(1.07)^3-1}=\frac{\$35\times 1.225043}{.225043}$

= $190.52.

The 3 power of 1.07 is found from Compound Interest Table.

(3.) $b = \$1000$, $r = .08$, $n = 4$.

$x=\frac{br(1+r)^n}{(1+r)^n-1}=\frac{\$1000\times\frac{8}{100}(1.08)^4}{(1.08)^4-1}=\frac{\$80\times 1.360489}{.360489}=$ $301.93.

$80 is the interest on $1000 for 1 *yr.*

1.360489 is found in compound interest table opposite 4 years at 8%.

COMMERCIAL DISCOUNT.

PAGE 196.

(1.) $D = b(1 - r) = \$3.75(1 - .10) = \$3.75 \times .90 = \$3.37\frac{1}{2}$.

To find the selling price when the cost and loss % are given.

RULE.—Multiply the cost by one minus the rate.

(2.) $D = b(1 - r)$; $b = \frac{D}{1 - r} = \frac{\$3.375}{1 - .10} = \frac{\$3.375}{.90} = \$3.75$.

To find the cost when the selling price and loss % are given.

RULE.—Divide the selling price by one minus the rate.

(3.) $D = b - br$; $br = b - D$; $r = \frac{b - D}{b} = \frac{\$3.75 - 3.37\frac{1}{2}}{\$3.75} = .10 = 10\%$.

To find the loss per cent when the buying and selling prices are given.

RULE.—Divide the difference between the cost and selling price by the cost.

REMARK.—The cost is the base of Profit and Loss.

BANK DISCOUNT.

PAGES 198-200.

(2.) $D = b(1 - rt)$; $b = \frac{D}{1 - rt} = \frac{\$200}{1 - .0204} = \frac{\$200}{.9796} = \$204.16$. $tr = .08 \times \frac{23}{365} = .0204$.

Since \$1 of the face of the note yields \$.9796, it will take as many dollars to yield \$200 as .9796 is contained times in 200, or \$204.16.

(3.) $D = b(1 - rt) = b - brt = \$450 - 450 \times .05 \times \frac{33}{365} = \$447.97.$

$\$450 \times .05 \times \frac{33}{365} = \$2.034$; $\$450 - 2.034 = \$447.97.$

$\$450 \times .10 \times \frac{33}{365} = \$4.068$; $\$450 - 4.068 = \$445.93.$

$\$450 \times .08 \times \frac{93}{365} = \$9.17$; $\$450 - 9.17 = \$440.83.$

(4.) $rt = .07 \times \frac{33}{365} = .006328$; $b = \frac{D}{1 - rt} = \frac{\$350}{1 - .006328} = \frac{\$350}{.99367} = \$352.22.$

$rt = .10 \times \frac{93}{365} = .0255$; $b = \frac{D}{1 - rt} = \frac{\$150}{1 - .0255} = \frac{\$150}{.9745} = \$153.93.$

$rt = .05 = \frac{63}{365} = .00863$; $b = \frac{D}{1 - rt} = \frac{\$750}{1 - .00863} = \frac{\$750}{.99137} = \$756.53.$

Since \$1 of the face of the note yields \$.99137 it will take as many to yield \$750 as .99137 is contained times in 750.

(5.) $rt = .10 \times \frac{33}{365} = .00904$; $b = \frac{D}{1 - rt} = \frac{\$300}{1 - .00904} = \frac{\$300}{.99096} = \$302.74.$

(6.) $D = b - rt = \$380 - 380 \times .08 \times \frac{24}{365} = \$380 - 2 = \$378.$

(7.) $brt = b - D$; $r = \frac{b - D}{bt} = \frac{\$101.76 - 100}{101.76 \times \frac{63}{365}} = \frac{\$1.76}{\$17.56} = .10.$

(8.) $brt = b - D$; $t = \frac{b - D}{br} = \frac{\$100 - 99.06}{\$100 \times .10} = \frac{.94}{10}$ $= \frac{94}{1000}\,yr.$ $\frac{94}{1000}$ of 365 *da.* = 34 *da.*

(9.) $t = \frac{b - D}{br} = \frac{100 - 987}{1000 \times .06} = \frac{13}{60}$. $\frac{13}{60}$ of 365 = 79 *da.*

(10.) \$175 × .10 × $\frac{63}{365}$ = \$3.02 discount, the interest on face of note. \$175 — \$3.02 = \$171.98, proceeds.

(11.) Int. on \$360, at 7%, from March 12, 1874, to June 15, 1875, (1 *yr.* 95 *da.*), is \$360 × .07 × $1\frac{95}{365}$ = \$31.76; \$360 + 31.76 = \$391.76.

\$391.76 — 50 = \$341.76, due June 15, 1875.

Int. on \$341.76, at 7%, from June 15, 1875 to Feb. 15, 1877 (1 *yr.* 244 *da.*), is \$341.76 × .07 × $1\frac{244}{365}$ = \$39.98; \$341.76 + 39.98 = \$381.74.

\$381.74 × .10 × $\frac{66}{365}$ = \$6.90 discount; \$381.74 — 6.90 = \$374.74.

\$381.34 becomes the amount to be discounted.

(*294.*) *To find the proceeds of a note made at Bank.*

RULE.—Find the interest on the face of the note for 3 days more than the given time and it will be the discount. Subtract the discount from the face of the note and the remainder will be the proceeds.

To find the face of a note at Bank when the proceeds, rate, and time are given.

RULE.—Divide the proceeds by the proceeds of \$1 for the given rate and time. The proceeds of \$1 is \$1 minus the product of the rate and time.

To find the avails of a note at Bank discount, when it bears interest, or when there are partial payments.

RULE.—Find the amount of the note at its maturity and divide this amount by the proceeds of $1 for the time and rate for which it is discounted.

(12.) $100 × .07 × $\frac{93}{365}$ = $1.78, discount.
$100 — 1.78 = $98.22.

(13.) $1 × .07 × $\frac{93}{365}$ = $.0178, discount of $1; $1 — .0178 = $.9822, proceeds.
250 ÷ .9822 = $254.53, face of note.

(14.) Int. on $280, at 8%, for 160 *da.*, is $9.82.
$280 + 9.82 = $289.82, Amt.
$289.82 — 100 = $189.82, New principal.
Int. on $189.82, at 8%, for 1 *yr.* 139 *da.*, is $20.97.
$189.82 + 20.97 = 210.79, Amt. at maturity.
Discount on $210.79, at 10%, for 66 *da.*, is $3.82.
$210.79 × .10 × $\frac{66}{365}$ = $3.82; $210.79 — 3.82 = $206.97.

(15.) Amt. of $500 for 1 *yr.* is $530; $530—30 = $500, July 23, 1874.
Amt of $500 for 1 *yr.* is $530; $100 × .06 × $\frac{114}{365}$ = $3.12.
$100 + 3.12 + 40 = $143.12; $530—143.12 = $386.88, due July 23, 1875.
$386.88 + .06 × $\frac{214}{365}$ = $18.125; $386.88 + 18.125 = $405.005.
Discount on $405.005 for 80 *da.*, at 7%, is $6.213.
$405.005 — 6.213 = $398.79, proceeds.

TRUE DISCOUNT.

PAGES **200–211.**

(1.) $350 × .07 × $2\frac{114}{365}$ = $66.18; $350 + 66.18 = $416.18, amt. for 2 *yr.* 256 *da.* at 7%; $1 × 10 × $1\frac{113}{365}$ = $.13096. Amount of $1 for 1 *yr.* 113 *da.* at 10% is $1.13096.
$416.18 ÷ 1.13096 = $367.99— avails.

(2.) 4 *yr.* 9 *mo.* 18 *da.* = 4.8 *yr.* $1 × .05 × 4.8 = $.24.
Amt. of $1 for 4.8 *yr.* at 5% is $1.24.
$1860 ÷ 1.24 = $1500, present worth.

(3.) $280 × .10 × $2\frac{145}{360}$ = $67.12.
Amt. of $280 for 2 *yr.* 145 *da.* at 10% is $347.12.
$1 × .08 × $\frac{262}{360}$ = $.057424.
Amt. of $1 for 262 *da.* at 8% is $1.057424.
$347.12 ÷ 1.057424 = $328.27, avails.

(4.) $300 × .10 × 3 = $90, interest.
$300 + 90 = $390, amt. at maturity.
$1 × .08 × 3 = $.24; $1 + .24 = $1.24, amt. of $1 for given rate and time.
$390 ÷ 1.24 = $314.52, present worth.

Since $1 amounts to $1.24 in 3 *yr.* at 8%, it will take as many dollars to amount to $390 as 1.24 is contained times in 390, or $314.52.

(5.) $300 × .08 × 3 = $72; $300 + 72 = $372, amt. of note.
$1 × .10 × 3 = $30; amt. of $1 in 3 *yr.* at 10% is $1.30.
$372 ÷ 1.30 = $286.15, avails.

The rate of discount being greater than the rate of interest, the avails are less than the face.

(6.) The present worth when the note was made = the face of the note, since the money was on interest for all money was worth.

Amt. of $300 in 3 *yr.* at 10% is $390.
Amt. of $1 for 2 *yr.* at 10% is $1.20.
$390 ÷ 1.20 = $325, avails.

Amt. of $1 for 6 *mo.* at 10% is $1.05.
$390 ÷ 1.05 = $371.43, avails.

(7.) Amt. of \$1 for 3 *yr.* at 10% is \$1.30.
\$300 ÷ 1.30 = \$230.77.
Amt. of \$1 for 2 *yr.* at 10% is \$1.20.
\$300 ÷ 1.20 = \$250.
Amt. of \$1 for 6 *mo.* at 10% is \$1.05.
\$300 ÷ 1.05 = \$285.71.

(8.) \$500 × .10 × $\frac{93}{365}$ = \$12.74, bank discount.
\$1 × .10 × $\frac{93}{365}$ = \$.02548; amt. of \$1 for 93 *da.* at 10% is \$1.02548.
\$500 ÷ 1.02548 = \$487.577, proceeds.
\$500 — 487.577 = \$12.42, true discount.

(9.) Amt. of \$800 for 1 *yr.* at 6% is \$848.
Amt. of \$848 for 1 *yr.* at 6% is \$898.88.
Time from Oct. 24 to March 18 is 4 *mo.* 24 *da.* = $\frac{4}{10}$ *yr.*
\$150 × .06 × .4 = \$3.60; amt. of \$150 for $\frac{4}{10}$ *yr.* at 6% is \$153.60.
\$898.88 — 153.60 = \$745.28, due March 18, 1869.
\$745.28 × .06 × $1\frac{3}{360}$ = \$45.089;
Amt. of \$745.28 for $1\frac{3}{360}$ *yr.* at 6% is \$790.369.
Time from Nov. 12 to March 21 is 4 *mo.* 9 *da.* = $\frac{43}{120}$ *yr.*
Amt. of \$240 for $\frac{43}{120}$ yr. at 6% is \$245.16, amt. of pay't.
\$790.369 — 245.16 = \$545.209, amt. at maturity.
Int. on \$545.209 for 77 *da.* at 10% is \$11.50, the discount.
\$545.209 — 11.50 = \$533.71, the proceeds.

(10.)

Amt. of \$750 for 1 *yr.* at 7% is	\$802.50	
Amt. of \$400 for 8 *mo.* 25 *da.* is	\$420.61	amt. of pay't.
From Aug. 12 to May 7 is 8 *mo.* 25 *da.*,	\$381.89	end of 2d yr.

\$381.89 — 250 = \$131.89, bal. at end of 3d yr.
Amt. of \$131.89 for 1 *yr.* at 7% is \$141.12, paid at end of 4th yr.
Now the note was discounted 86 days before end of 3d

yr. Int. on \$381.89 for 1 *yr.* is \$26.73; \$250 + 26.73 = \$276.73, amt. to be discounted for 86 *da.* at 10%; \$1 × .10 × $\frac{86}{365}$ = \$.02356; \$276.73 ÷ 1.02356 =

Pro. of \$276.73 for 86 *da.* at 10% = \$270.36
Pro. of \$141.12 for 1 *yr.* 86 *da.* at 10% = \$125.60
Value of the note at time of sale, \$395.96 avails.

(17.) Amt. of \$2500 for 5 *yr.* at 7% is \$3375.

$$x = \frac{A}{1 + rt + r^2(1 + 2 + 3 + \text{etc. to } t - 1)} =$$

$$\frac{\$3375}{1 + \frac{1}{10} \times 5 + \frac{1}{100}(1 + 2 + 3 + 4)} = \frac{\$3375}{1.60} = \$2109.375,$$

present worth at deferred annual interest.

Amt. of \$1 from table, at compound interest for 5 *yr.* at 10%, is \$1.61057.

\$3375 ÷ 1.61051 = \$2095.61, P. W. at comp. int.

Amt. of \$1 for 5 *yr.* at 10% simple interest is \$1.50.

\$3375 ÷ 1.50 = \$2250, P. W. at common discount.

Int. on \$3375 for 5 *yr.* at 10% is \$1687.50.

\$3375 — 1687.50 = \$1687.50, P. W. at bank discount.

(18.) Amt. of \$1000 for 20 *yr.* at 5% = \$2000.

$$\frac{\$2000}{1 + \frac{1}{10} \times 10 + \frac{1}{100}(1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9)}$$

$$= \frac{\$2000}{1 + 1 + \frac{45}{100}} = \frac{\$2000}{2.45} = \$816.32.$$

Time of discount is 10 *yr.*

PAGE 209.

(21.) Int. on \$2500 for 1 *yr.* at 8% is \$200.

\$200 + 500 = \$700, first payment.

Int. on \$2000 for 1 *yr.* at 8% is \$160.

\$800 + 160 = \$960, 2d payment.

Int. on \$1200 for 1 *yr.* at 8% is \$96.

\$1200 + 96 = \$1296, 3d payment.

$t = 3$, $r = .10$, $a = \$700$, $b = \$960$, $s = \$1296$.

$$x = \frac{(1+r)^{t-1}\,a + (1+r)^{t-2}\,b + s}{(1+r)^t} =$$

$$\frac{(1.10)^2 \times 700 + (1.10) \times 960 + 1296}{(1.10)^3} = \frac{\$3199}{1.331} =$$

\$2403.46, present worth.

STOCKS AND BONDS.

PAGES 217-220.

(3.) $44\% + \frac{1}{8} = 44\frac{1}{8}\%$; $.44\frac{1}{8} \times 20000 = \$8825$.

Since \$1 in bonds costs $\$.44\frac{1}{8}$, \$20000 will cost 20000 times $\$.44\frac{1}{8}$, or \$8825.

(4.) $\$.07 \div 1.09 = .0642$, or 6.42%.

Since \$1 in bonds yields \$.07, and \$1 in bonds costs \$1.09, \$1 investment yields $.07 \div 1.09$, or .07. Hence the rate per cent is 7.

(5.) $5\% \div .84\frac{1}{2} = 5\frac{155}{169}\%$.

(6.) $5\% \div .96\frac{1}{2} = 5\frac{35}{193}\%$.

Since $\$.96\frac{1}{2}$ investment yields 5*c.*, \$1 investment yields as many cents as $.96\frac{1}{2}$ is contained times in 5*c.*, or $5\frac{35}{193}c.$, *i. e.*, $5\frac{35}{193}\%$.

(7.) Amt. of \$1 at 5% compound int. for 4 *yr.*, is \$1.215506.

Amt. of \$1 at 5% compound int. for 10 *yr.*, is \$1.628895.

Amts. of \$100 for same times and rate are \$121.55 and \$162.889. Hence France paid 162.8895 *fr.* for $84\frac{1}{2}$, and Germany 121.5506 for $96\frac{1}{2}$.

$\frac{162.8895}{84\frac{1}{2}}$ is to $\frac{121.5506}{96\frac{1}{2}}$ as 193 is to 125, nearly.

(8.) \$100 × 100 = \$10000; \$10000 × .00½. = \$20, brokerage.

100% − ⅒% − 99.9%.

\$77⅜ × 100 = \$7737.50; 7737.50 + 20 = \$7757.50.

\$7757.50 ÷ .999 = \$7765.26; stamp added gives \$7765.28.

Exchange is reckoned on amt. sent to N. Y., which includes brokers' fees.

(9.) \$3384.40 ÷ 1.001 = \$3381, amt. in N. Y.

One share costs \$60.25 + .125 = \$60.375.

\$3381 ÷ \$60.375 = 56 shares.

Since 1 share costs \$6.375, \$3381 will buy as many shares as 60.375 is contained times in 3381, or 56 shares.

(10.) 3½% + 3½% = 7%; \$.07 × 17500 = \$1225, dividends.

Since \$1 yields 7*c.*, \$17500 yields 17500 times 7*c.*, or \$1225.

(11.) \$500 × 15 = \$7500, par value.

\$.125 × 7500 = \$937.50, dividend.

Since \$1 yields 12½*c.*, \$7500 yields 7500 times 12½*c.*, or \$937.50.

(12.) \$100 × 54 = \$5400, par value.

\$.08 × 5400 = \$432, dividend.

(13.) 10% ÷ 1.105 = 9$\frac{11}{121}$%.

Since \$1.105 investment yields 10%, \$1 investment yields as many % as 1.105 is contained times in 10, or 9$\frac{11}{121}$%.

(15.) 7% ÷ 10% = .7 of \$1, or 70 cents.

Since \$1 investment is to yield 10*c.* \$$\frac{1}{10}$ investment must yield 1*c.*, and \$$\frac{7}{10}$ investment 7*c.*

(16.) \$35000 − 3000 = \$32000, amt. divided.

\$32000 ÷ \$48000 = .06⅔, or 6⅔%.

(17.) \$1.08 × 5000 = \$5400.

\$1.07$\frac{3}{4}$ + $\frac{1}{4}$ = \$1.08, cost of \$1 worth of stock.

\$5400 × 1.002 = \$5410.80; add stamp = \$5410.82.

50 shares = \$5000, par value. Now since \$1 worth of stock costs \$1.08, \$5000 worth of stock will cost 5000 times \$1.08, or \$5400.

(18.) 98$\frac{3}{4}$ + $\frac{1}{4}$ = 99; \$15000 ÷ \$99 = 151 shares and \$51 remaining.

\$15000 — 51 = \$14949, cost of shares.

\$.00$\frac{1}{8}$ × 14949 = \$18.686, exchange; add 2*c.* per stamp = \$18.706.

\$51 — 18.706 = \$32.294.

Paying the cost of exchange out of the \$51, he has left \$32.294. Since \$1 cost 99*c.*, \$100, or 1 share, cost \$99.

(19.) Amt. of \$4 for 6 *mo.* at 10% is \$4.20; \$4.20 + \$4 = \$8.20.

\$.016 × 50 = \$.80; \$8.20 — .80 = \$7.40.

7.40 ÷ 1.09 = .06$\frac{86}{100}$, or 6$\frac{86}{100}$%. Since an investment of \$109 pays \$7.40, \$1 pays $\frac{1}{109}$ of \$7.40, or .06$\frac{86}{100}$.

(20.) \$91$\frac{1}{4}$ × 200 = \$18250, cost of N. Y. C.

\$76$\frac{7}{8}$ × 200 = \$15375, cost of C. and N. W.

\$33625, entire cost.

\$94$\frac{5}{8}$ × 200 = \$18925

\$83$\frac{3}{8}$ × 200 = \$16675

Amt. received = \$35600.

\$.00$\frac{1}{4}$ × 400 = \$100, brokerage. \$35600 — \$100 = \$35500, net. \$35500 — \$33625 = \$1875, gain.

DOMESTIC EXCHANGE.

PAGES **223–227.**

(1.) \$1.00$\frac{1}{10}$ × 2500 = \$2502.50.

Since \$1 in exchange cost \$1.00$\frac{1}{10}$, \$2500 cost 2500 times \$1.00$\frac{1}{10}$, or \$2502.50.

(2.) $\$1.00\frac{1}{12} \times 1500 = \$1501.25$.

(3.) $\$1.000 - .00\frac{1}{20} = \$.9995$. $\$.9995 \times 1825 = \$1824.088$.
Since \$1 sells for \$.9995, \$1825 sells for 1825 times \$.9995, or \$1824.088.

(4.) $\$2500 \div 1.001 = \$2497.502$.
Since \$1 buys 1.001, it takes as many dollars to buy \$2500 in draft, as 1.001 is contained times in 2500.

(5.) $\$1.001 \times 2497.502 = \$2500$.
Since \$1 sells for \$1.001 \$2497.502 sells for 2497.502 times \$1.001, or \$2500.

(8.) $\$1.00 + .02 = \$1.02$; $\$.05 \times \frac{63}{365} = \frac{63}{73}cts.$, the time discount.
$\$1.02 - .00\frac{63}{73} = \$1.01\frac{10}{73}$, cost of \$1 of exchange.
$\$1.01\frac{10}{73} \times 750 = \$758.53$, cost of \$750 exchange, besides 2*c.* stamp.
$\$1.01\frac{10}{73} \times 85.50 = \$86.47$, cost of \$85.50 exchange, besides stamp.

(9.) Time discount on \$1 is $4c. \times \frac{243}{365}$, or $1\frac{7}{365}c$. To this adding the exchange, we have total discount on \$1, $1\frac{7}{365}c. + 1\frac{1}{2}c. = 2\frac{371}{730}$. Hence \$1 exchange costs \$1.00 — $\$0.02\frac{371}{730}$. or $\$0.97\frac{359}{730}$.
Cost of \$500, 500 times $\$0.97\frac{359}{730}$, or \$487.40, without stamp.
Cost of \$184.25, 184.25 times $\$0.97\frac{359}{730}$, or \$179.63, including 2*c.* for stamp.

(10.) Time discount $\frac{243}{365}$ of 5*c.*, or $\$0.00\frac{63}{73}$. Hence,
$\$1.01\frac{3}{4} - \$0.00\frac{63}{73} = \$1.01\frac{817}{1095}$, or $100\frac{817}{1095}\%$.
$\$1.01\frac{3}{4} - \$0.00\frac{63}{73} = \$1.00\frac{256}{365}$, or $100\frac{256}{365}\%$.
$\$1.01\frac{3}{4} - \$0.00\frac{63}{73} = \$1.00\frac{139}{365}$, or $100\frac{139}{365}\%$.
$\$1.01\frac{3}{4} - \$0.00\frac{1}{73} = \$1.01\frac{187}{1095}$, or $100\frac{187}{1095}\%$.

(11.) $4.5 \times \frac{33}{365} = \frac{297}{730}$; $98.5 - \frac{297}{730} = 98\frac{34}{365}$.
$4.5 \times \frac{63}{365} = \frac{567}{730}$; $98.5 - \frac{567}{730} = 97\frac{234}{365}$.
$4.5 \times \frac{93}{365} = 1\frac{197}{730}$; $98.5 - 1\frac{197}{730} = 97\frac{128}{365}$.
$4.5 \times \frac{13}{730} = \frac{117}{730}$; $98.5 - \frac{117}{730} = 98\frac{124}{365}$.

(13.) $\$0.05 \times \frac{93}{365} = \$0.00\frac{20}{73}$; $\$1.00 - \$0.01\frac{20}{73} = \$0.98\frac{53}{73}$.
$\$0.98\frac{53}{73} \times 998 = \$985.29$.

(14.) $\$1259.375 \div \$1.0075 = \$1250$. Since $\$1$ face of draft sold for $\$1.0075$.

(15.) It means that a draft on N. Y. for $\$1000$ can be bought for $\$999.25$; and as the discount on $\$1000$ is $75c.$, the discount on $\$1$ is .075 cents, or $\frac{3}{40}\%$.
$\$0.05 \times \frac{93}{365} = \$0.01\frac{20}{73}$; $1\frac{20}{73}\% + \frac{3}{40}\% = 1\frac{101}{292}\%$; $100\% - 1\frac{101}{292}\% = 98\frac{191}{292}\%$. $2466.28 \div .98\frac{191}{292} = \$2500$, face of draft.

(16.) $\$16 \times 100 = \$1600$; $\$1600 \times .60 = \$960$.
$\frac{1}{8}\% = \frac{146}{730}\%$; $4\frac{1}{2}\% \times \frac{40}{365} = \frac{180}{365} = \frac{36}{73}\%$.
$\frac{328}{730} - \frac{146}{730} = \frac{182}{730}\%$. Time discount minus the rate of exchange gives exchange discount.
$99\frac{730}{730}\% - \frac{182}{730} = 99\frac{548}{730}\%$; $\$.99\frac{548}{730} \times 960 = \$957.21$, including stamp.

FOREIGN EXCHANGE.

PAGES 229, 230.

(4.) $\$40\frac{3}{4} \times 5000 = \$2037.50$ in gold.
$\$2037.50 \times 1.10 = \$2241.25$ in currency.
$\$1.00\frac{1}{8} \times 2241.25 = \$2245.73$, whole cost.
Since 1 guilder cost $40\frac{3}{4}c.$, 5000 guilders cost 5000 times $40\frac{3}{4}c.$, or $\$2037.50$ gold.
Since $\$1$ in gold is worth $\$1.10$ in currency, $\$2037.50$ in gold are worth 2037.5 times $\$1.10$.

(5.) \$4.88 × 1.12 = \$5.4656, value of £1 in currency.
\$1.00¼ × 5.4656 = \$5.479264, cost of £1.
\$5000 at \$5.479264 to the pound = £912 10*s.* 9½*d.*

(6.) $\frac{\$1.00}{5.14}$ = 19.455 *cts.* cost of 1 *fr.*; hence \$.19455 — \$.193 = \$.00155, percentage, or premium on 1 *fr.*
.00155 ÷ .193 = .00$\frac{155}{193}$. Percentage divided by base = rate.

\$.4075 — .39 = \$.0175; .0175 ÷ .39 = .04$\frac{11}{39}$, or 4$\frac{11}{39}$%.
Par value of 1 guilder is 39 cts. (See page 228.)

\$4.88 — 4.8665 = \$.0135; .0135 ÷ 4.8665 = .00$\frac{2700}{4733}$, or $\frac{2700}{4733}$%.

(7.) \$4.8665 × 1.02 = \$4.96, the quotation.
.193 × 1.0175 = .1963775, cost of 1 *fr.*
\$1.00 ÷ \$1963775 = 5.092, quotation.
92.5 ÷ 1.025 = 97.58, quotation.

(8.) 495 ÷ 96.12 = 5.15 *fr.* Since \$96.12 buys 495 *fr.*, \$1 buys as many *francs* as 96.12 is contained times in 495.

(9.) \$566.50 ÷ 1.1025 = \$513.83, gold value.
As 2155 *marks* cost \$513.83, 1 *mark* cost $\frac{\$513.83}{2155}$, and 4 *marks* $\frac{\$513.83 \times 4}{2155}$, or \$0.95⅝. Hence the quotation is 95⅝.

(10.) 13550 *M.* ÷ 4 = 3387.5 times 4 *marks*.
\$.94¼ × 3387.5 = \$3193.718, cost of 13550 *marks* in gold.
\$1.00⅛ × 3193.718 = \$3197.71, gold including brokerage.
\$3500.40 ÷ \$3197.71 = 1.09½. Hence the quotation is 109½.

(11.) 40 ÷ 1.02 = 39$\frac{1}{51}$, gold value of guilder.

EQUATION OF PAYMENTS.

PAGES 234, 235.

(3.) As the payments bear interest for all money is worth, it is immaterial when they are made, as the cash is no better than the obligation. Hence the equated time is *any time*.

(5.)

$300	matures July 4.	To Nov. 1 is 120 *da.* Int. at 10%	=	$9.86
$500	" Sept. 2.	To Nov. 1 is 60 *da.* Int. at 10%	=	$8.20
$750	" Nov. 1.	Total int.,		$18.06

On $1550 for 1 *da.* int. at 10% is 42.4 *cts.*

18.06 ÷ .424 = 43. Hence the equated time is 43 *da.* antecedent to Nov. 1, which is Sept. 19/22, 1876.

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(8.)

$420	matures Feb. 2.	To Sept. 15 is 225 *da.* Int. at 10%	=	$25.89
$420	" May 20.	To Sept. 15 is 118 *da.* Int. at 10%	=	$13.57
$910	" Sept. 15.	Total int.,		$39.46

Int. on $1750 for 1 *da.* at 10% is 47.9 *cts.*

$39.46 ÷ $0.479 = 82. Hence the equated time is 82 *da.* antecedent to Sept. 15, and by the table we find it to be June 25.

(9.)

$860	matures July 5th,	To Dec. 10th is 158 *da.* Int. at 10%	=	$37.23
$480	" Oct. 20th,	To Dec. 10th is 51 *da.* Int. at 10%	=	6.70
$760	" Dec. 10th,	Total int.,		$43.93

Int. on $2100 for 1 *da.*, at 10%, is 57.3 *cts.*

43.93 ÷ .573 = 76. Hence the equated time is 76 *da.* antecedent to Dec. 10, which from the table is Sept. 25th.

(10.) $200 on interest for 4 *mo.*, at 6%, yields $4
$200 on interest for 2 *mo.*, at 6%, yields $2
$200 on int. for 1 *mo.*, at 6%, yields $1. $6

$6 ÷ $1 = 6; hence the equated time is 6 *mo.*; and the remaining $200 must be on int. for 6 *mo.* to compensate for the payments of the other sums before they become due.

AVERAGING ACCOUNTS.

PAGES 236–238.

(1.)

Int. on $400 for 276 *da.* at 10%	=	$30.94	Int. on $350 for 250 *da.* at 10%	=	$23.79
" " 600 " 116 " " "	=	19.07	" " 600 " 179 " " "	=	20.42
" " 1000 " 47 " " "	=	12.88	" " 250 " 141 " " "	=	9.66
" " 500			" " 500 " 122 " " "	=	18.08
$2500	Int. on debits,	$62.19	$1700	Int. on credits,	$81.13
1700					62.19
Int. on $800 per 1 day at 10% is 21.9*c*.			Balc. of int. in favor of debtor,		$18.94.

$18.94 ÷ $0.219 = 86. Hence the equated time is 86 days after March 8, which is June 2.

(2.) Int. on $500 for 1 *da.* at 10% is 13.7*c*.

$28.60 ÷ $.137 = 208; or 208 *da.*, or Nov. 20.

From Nov. 20 to Dec. 1 is 11 *da.*; 208 *da.* + 11 = 219 *da.*

Amt. of $500 for 219 *da.* at 6% is $518.

(3.)

Maturity of Debits.	*Time.*	*Debit Int.*	*Maturity of Credits.*	*Days before Latest Maturity.*	*Credit Int.*
Apr. 30, $600,	122,	$20.06	Apr. 20, $400,	132,	$14.46
June 5, 200,	86,	4.71	July 10, 600,	51,	8.38
June 10, 800,	81,	17.75	Aug. 10, 500,	20,	2.74
July 15, 700,	46,	8.82	Aug. 30, 500,		
$2300		$51.34	$2000		$25.58
2000		25.58			
$300		$25.76			

Int. on $300 at 10% for 1 *da.* is 8.22*c*.

$25.76 ÷ $.0822 = 313; 313 *da.* anterior to Aug. 30, or 334 *da.* anterior to Sept. 20.

Amt. of $300 at 6% for 334 *da.* is $316.47.

(4.)

Maturity of Debits.		Days before Latest Maturity.	Debit Int.	Maturity of Credits.		Days before Latest Maturity.	Credit Int.
Nov. 8,	$236.17,	164,	$10.611	Nov. 3,	$439.27,	169,	$20.34
Dec. 1,	819.63,	141,	31.662	Nov. 25,	213.16,	147,	8.58
Jan. 4,	142.13,	107,	4.166	Jan. 13,	100.00,	98,	2.68
Mar. 13,	947.22,	39,	10.121	Feb. 24,	262.18,	56,	4.02
Apr. 8,	1050.00,	13,	3.74	Mar. 30,	327.48,	22,	1.97
	$3195.15		$60.30	Apr. 21,	520.75		
	1862.84		37.59		$1862.84		$37.59
	$1332.31		$22.71 ÷ $.365 = 62, or 62 *da.*				

Int. on $1332.31 for 1 *da.* at 10% is 36.5*c.*

Now 62 *da.* anterior to Apr. 21 is Feb. 18, which is 48 days after Jan. 1, 1873. The time for $1332.31 to be paid is 48 after Jan. 1. Hence we find the present worth of $1332.31 for 48 *da.* at 5%, which is $1323.61*c.*

(5.) This account stands thus:

WHITE & Co. in % with HALL & JONES.

1875.	DEBITS.		
Sept. 23.	Frt. on 2000 *bu.* wheat @ 5*c.*,	$100	
	Insurance on same @ 1*c.*,	20	$120.00
Oct. 4.	Frt. on 2500 *bu.* wheat @ 5*c.*,	$125	
	Insurance on same @ 1*c.*	25	150.00
Nov. 18.	Frt. on 3000 *bu.* wheat @ 5*c.*,	$150	
	Insurance on same @ 1*c.*,	30	180.00
Nov. 20.	Storage of 2000 *bu.* wheat 2 *mo.*,	$80	
	Com. for selling same @ 1¼*c*,	35	115.00
Dec. 25.	Storage of 3000 *bu.* wheat 40 *da.*,	$90.00	
1876.	Com. for selling same @ 1¼*c.*,	52.50	142.50
Jan. 1.	$3000 Draft at $\frac{1}{10}$% premium,		3003.00
Jan. 16.	St'ge of 2500 *bu.* wheat 3½ *mo.*,	$162.50	
	Com. for selling same at 1¼*c.*,	43.75	206.25
	Total debits,		$3916.75

1875.	CREDITS.	
Nov. 20.	Cash for 2000 *bu.* wheat sold, $1.75	$3500.00
Dec. 25.	" " 3000 " " " 1.70	5100.00
1876.		
Jan. 16.	" " 2500 " " " 1.80	4500.00
	Total credits,	$13100.00
		3916.75
	Excess of credits,	$9183.25

In averaging this account, as the storage and commission for selling were paid at the dates of selling, it will save a little labor to omit these items, deducting them from the credits of the same dates, thus leaving the credit for Nov. 20, $3500—$115 = $3385; for Dec. 35, $5100 — $142.50 = $4957.50; and for Jan. 16, $4500 — $206.25 = $4293.75.

We then have, reckoning to Jan. 16, 1876,

DEBITS.

$120,	115 *da.*,	$1.20	× 115* =	$138.00
150,	104 *da.*,	1.50	× 104 =	156.00
180,	59 *da.*,	1.80	× 59 =	106.20
3003,	15 *da.*,	30.03	× 15 =	450.45
			Debit interest,	$850.65

CREDITS.

$3385,	57 *da.*,	$33.85	× 57 =	$1929.45
4957.50,	22 *da.*,	49.575	× 22 =	1090.65
4293.75,	0 *da.*			
			Credit interest,	$3020.10
				850.65
			Excess of debit interest,	$2169.45

Interest on $9183.25 for 1 *da.* at 1%, $91.8325. Hence 2169.45 ÷ 91.8325 = 24 (nearly), is the number of days

* Same as reckoning 1% per day, as the rate is immaterial. See text-book 329 (*a*), p. 232.

anterior to Jan. 16, 1876, at which the balance of the acct., $9183.25, was due, or the equated time. This is Dec. 23, 1875. Now the amt. of $9183.25, at 7% from Dec. 23 to Mar. 1, 68 *da.*, is $9303.01, the *amt.* due at the latter date.

PROPERTY INSURANCE.

Pages 240, 241.

(1.) $5000 + $2000 + $2000 = $9000; and $\frac{2}{3}$ of $9000 is $6000, the amt. insured.

$\frac{3}{4}$% of $6000 = $22.50, annual premium.

$1500 + $600 + $300 = $2400, rec'd from company.

(2.) $1700 + $200 + $100 = $2000, amt. insured.

$18 ÷ 3 = $6 per year.

$6 ÷ $2000 = .00$\frac{3}{10}$, or $\frac{3}{10}$%.

(3.) $145.60 ÷ .01$\frac{4}{5}$ = $8736.

$8736 ÷ $\frac{3}{4}$ = $11648.

(4.) $89.10 ÷ .01$\frac{3}{8}$ = $6480.

(5.) $1.50 × 3000 = $4500; 100% − 3$\frac{1}{2}$% = 96$\frac{1}{2}$%.

$4500 ÷ .965 = $4663.21.

Since 3$\frac{1}{2}$% of the policy is premium, 96$\frac{1}{2}$% of the policy is to cover property; hence to cover $4500 worth of property it will take a policy of as many dollars as .965 is contained times in 4500.

(6.) $50 − $2 = $48 premium.

Since the premium is $48 for 3 *yr.*, for 1 *yr.* it is $\frac{1}{3}$ of $48, or $16.

$\frac{1}{2}$ of $8000 = $6000; $16 ÷ $6000 = .002$\frac{2}{3}$, or .0$\frac{2}{3}$%.

(7.) $\frac{4}{5}$ of \$6000 = \$4500; $\frac{1}{5}$% of \$4500 is \$27.
\$27 + \$2 = \$29 first payment.

Amt. of first payment,	\$29,	for $6\frac{1}{4}$ *yr.* at 7%	=	\$41.69
" " 2d "	\$27,	" $5\frac{1}{4}$ *yr.* " "		36.92
" " 3d "	\$27,	" $4\frac{1}{4}$ *yr.* " "		35.032
" " 4th "	\$27,	" $3\frac{1}{4}$ *yr.* " "		33.142
" " 5th "	\$27,	" $2\frac{1}{4}$ *yr.* " "		31.252
" " 6th "	\$27,	" $1\frac{1}{4}$ *yr.* " "		29.362
" " 7th "	\$27,	" $\frac{1}{4}$ *yr.* " "		27.472
				\$234.87

\$6000 — \$4500 = \$1500.
\$1500 + \$234.87 = \$1734.87 whole loss.

(8.) \$37.50 ÷ .015 = \$2500, amt. of policy.
\$2500 ÷ $\frac{4}{5}$ = \$3750, value of barn and contents.

(9.) \$.125 ÷ \$6.25 = .02, or 2%.
Percentage divided by base gives rate.

(10.) \$17.50 ÷ .015 = \$1166.66$\frac{2}{3}$, face of policy.
\$1166.66$\frac{2}{3}$ — 17 50 = \$1149.16$\frac{2}{3}$, value of property.

CERTAIN ANNUITIES.

PAGES 243–247.

(4.) $P_{\mu} = \frac{5(2 + (10 - 1).06)}{1.60} \times \$500 = \frac{5(2.54)}{1.60} \times \$500 = \$3968.75$.

Since \$500 is to be paid at the end of each year for 10 years, we have \$500 + int. on \$500 for 9 *yr.*, + \$500 + int. on \$500 for 8 *yr.*, + \$500 + int. on \$500 for 7 *yr.*, etc. for 10 payments, which = \$6350. P_{μ} of \$6350 for 10 *yr.* at 6% simple interest is \$3968.75; and \$3968.75 — \$3500 = \$468.75.

(7.) $P_{\overline{10}|} = \frac{(1.10)^{10} - 1}{.10(1.10)^{10}} \times \$500 = \frac{1.593742 \times \$500}{.10 \times 2.593742} =$ \$3072.28.

From table of compound interest we find $(1.10)^{10} =$ 2.593742.

Since the annuity bears 10% compound interest, we have the amount at compound int. of \$500 for 9 *yr.* + the amount of \$500 for 8 *yr.* + the amt. at comp. int. of \$500 for 7 *yr.*, etc., to the 10th payment, which is \$500. Now the sum of these amts. will be the value of the annuity 10 *yr.* hence, and the $P_{\overline{10}|}$ of the annuity will be the $P_{\overline{10}|}$ of this sum for 10 years at comp. int.

(8.) $A = \frac{(1+r)^t - 1}{r} a = \frac{(1.05)^{21} - 1}{.05} \times 100 =$ $\frac{2.785963 - 1}{.05} \times 100 =$ 3571.93, amt. of the annuity at 21 *yr.* of age.

$(1.05)^{10} \times (1.05)^{10} \times (1.05) = (1.05)^{21} = 2.785963.$

(9.)
$$A = \$27.1521 \times 500 = \$13576.05$$
$$A = \$18.5989 \times 500 = \$9299.45$$
$$A = \$23.2760 \times 500 = \$11638.$$
$$P_{\overline{10}|} = \$5.0757 \times 500 = \$2537.85$$
$$P_{\overline{10}|} = \$7.1078 \times 500 = \$3553.90$$
$$P_{\overline{10}|} = \$12.4622 \times 500 = \$6231.10.$$

(10.) $\$.20 \times 365 = \$73.$

$A = \frac{(1.06)^{45} - 1}{.06} \times 73 = \frac{13.76462 - 1}{.06} \times 73 =$ 15530.29.

Take the 10th power of 1.06 from comp. int. table, and we have $(1.06)^{10} \times (1.06)^{10} = 3.207136 = (1.06)^{20}$.

$(1.06)^{40} = (1.06)^{20} \times (1.06)^{20} = 10.285723.$
$(1.06)^{45} = (1.06)^{40} \times (1.06)^{5} = 10.285723 \times 1.338226 = 13.764621.$
$\$12.76462 \times 73 = \$931.8172.$
$\$931.817 \div .06 = \$15530.29.$

PERPETUITIES.

Pages 247, 248.

(2.) £1000 ÷ .03 = £33333 6*s.* 8*d.*

(3.) $\$2500 \div .04 = \$62500.$

ANNUITIES IN REVERSION.

Pages 248, 249.

(1.) $\$3.1463 \times 450 = \$1415.835.$
$\$5625 - \$1796.715 = \$3828.285.$

(2.) The present worth of \$1 annuity for 19 *yr.* is \$13.1339, and for 12 *yr.* \$9.3851, as we see from the table. Hence the present worth of \$1 for the given time is $\$13.1339 - \$9.3851 = \$3.7488$; and of \$90, $\$3.7488 \times 90 = \$337.39.$

(3.) Present worth of \$1 for 25 *yr.* at 6% $= \frac{(1.06)^{25} - 1}{.06(1.06)^{25}}$ $= \$12.78336.$
$(1.06)^{25} = (1.06)^{20} \times (1.06)^{5} = 3.207136 \times 1.338226.$
$P_{\overline{10}|}$ of \$1 for 10 *yr.* is \$7.3601.
$\$12.78336 - 7.3601 = \$5.42326.$
$\$5.42326 \times 1000 = \$5423.26$, set apart for son.
$\$12000 - 5423.26 = \$6576.74$, bequeathed to wife.

(4.) $P_{\overline{n}|} = \frac{a}{r(1+r)^t} = \frac{\$300}{.05(1.05)^{20}} = \frac{\$300}{.05 \times 2.653297}$ = \$2261.34.

(5.) $P_{\overline{n}|} = \frac{a}{r(1+r)^t} = \frac{\$360}{.05(1.05)^{10}} = \$4420.17.$

ANNUITIES IN ARREARS.

PAGE **249.**

(2.) $A = \frac{(1.05)^6 - 1}{.05} \times \$500 = \$3400.96$, amt. for 6 *yr.*

Or, by the table (***339***), \$6.8019 × 500 = \$3400.95.

$P_{\overline{n}|}$ of \$500 annuity for 9 *yr.* from the table (***340***) is \$7.1078 × 500 = \$3553.90. Hence the son receives \$3400.95 + \$3553.90 = \$6954.85.

SINGLE LIFE ANNUITIES.

PAGES **252–255.**

(2.) From the table we find that the *favorable opportunities* are 79, and the *whole number of opportunities* 847; *i. e.*, that of 847 persons who are living at 90, 79 will survive to 93; hence the mathematical probability that any one of the 847 will live to be 93 is $\frac{79}{847}$. (See ***347*** in the text-book.)

As above we have $\frac{88787}{89804}$, and $\frac{83383}{89810}$, the number of favorable opportunities, *i. e.*, the number of survivors at the latest age, divided by the whole number of opportunities, *i. e.*, the whole number taken at the younger age.

(3.) Same as above, we have $\frac{88848}{89804}$, $\frac{87817}{89804}$, $\frac{85582}{89804}$, $\frac{14474}{89804}$.

(7.) This right of dower is the same as a single life annuity of 4% on $\frac{1}{3}$ of \$45000, or 4% of \$15000; *i. e.*, an

annuity of $600. Finding from the table (*348*) the present worths of $1 for 1 *yr.*, 2 *yr.*, 3 *yr.*, etc., we have the following operation:

$$\begin{aligned}
\$0.9615 \times 600 \times \tfrac{4183}{5485} &= \$441.01 \\
0.9246 \times 600 \times \tfrac{3078}{5485} &= 311.41 \\
0.8890 \times 600 \times \tfrac{2148}{5485} &= 208.69 \\
0.8548 \times 600 \times \tfrac{1402}{5485} &= 131.09 \\
0.8219 \times 600 \times \tfrac{847}{5485} &= 76.15 \\
0.7903 \times 600 \times \tfrac{462}{5485} &= 39.94 \\
0.7599 \times 600 \times \tfrac{216}{5485} &= 17.96 \\
0.7307 \times 600 \times \tfrac{71}{5485} &= 6.31 \\
0.7026 \times 600 \times \tfrac{21}{5485} &= 1.61 \\
0.6756 \times 600 \times \tfrac{3}{5485} &= 0.22 \\
\text{Present worth,} & \quad \$1234.39
\end{aligned}$$

For further explanation see text-book, Ex. 5, p. 252, explanation. In solving it will save labor to omit the division by the common divisor, 5485, in the separate cases, and divide the sum of the products.

JOINT LIFE ANNUITIES.

PAGES 255, 256.

(2.) $\frac{88842}{89804} \times \frac{80353}{81050} = \frac{5531661}{6660206}$, nearly.
$\frac{11814}{11837} \times \frac{56871}{57017} = \frac{6181884}{6365357}$, nearly.

(3.) $\frac{44818}{87441} \times \frac{11814}{11837} \times \frac{11333}{11333} \times \frac{10311}{11111} \times \frac{81761}{81761} = \frac{85100103118}{88811465550}$.

(4.) $\$500 \div 1.50 = \$333\frac{1}{3}$ P., simple interest, without contingency.
$\frac{81333}{87441} \times \frac{74173}{78108} = \frac{6006828}{6831848}$, nearly.
$\frac{6006828}{6831848} \times \frac{1000}{3} = \$303.14$.

$.6209 × 500 = $310.45, $P_{\cdot}$ at comp. int., without contingency.

$\frac{1066198}{1073418} \times 310.45 = \$282.33.$

(5.) $200 ÷ 1.035 = $193.24, $P_{\cdot}$ for 1 *yr.*, without contingency.

$200 ÷ 1.071225 = $186.70, $P_{\cdot}$ for 2 *yr.*, without contingency.

$200 ÷ 1.108718 = $180.38, $P_{\cdot}$ for 3 *yr.*, without contingency.

The remainder of the work is in the text.

(6.) *At compound interest* the present worths of these three payments are, as found from table (*348*), $961.50, $924.60, and $889.00, without contingency. As explained in solution of Ex. 1, p. 255, the corresponding contingencies are $\frac{462}{847} \times \frac{79}{216} = \frac{36498}{182952}$, $\frac{216}{847} \times \frac{21}{216} = \frac{4536}{182952}$, and $\frac{79}{847} \times \frac{3}{216} = \frac{237}{182952}$. Hence we have the present worths of these 3 payments subject to the contingency of the survival of both persons:

$\$961.50 \times \frac{36498}{182952} + \$924.60 \times \frac{4563}{182952} + \$889 \times \frac{237}{182952}$,

or $\frac{\$961.50 \times 36498 + \$924.60 \times 4563 + \$889 \times 237}{182952} =$

$215.89, the present worth at compound interest.

At simple interest we have the present worths of the 3 payments *without contingency*, $961.50, $925.925, and $892.857. Hence, subject to the contingency, we have

$\frac{\$961.50 \times 36498 + \$925.925 \times 4563 + \$892.857 \times 237}{182952} =$

$215.94.

LIFE INSURANCE.

PAGES 259–264.

(2.) For the *rationale* see preceding example in the text-book. Since the denominators of the several fractions expressing the contingencies are the same, it will save labor to multiply the several present worths without

contingency by the numerators, and divide the sum of the products of the common denominator. The operation is as follows:

$.9524	× 3000	× 2476	=	$7074427.2
.9070	× 3000	× 2431	=	6614751.0
.8638	× 3000	× 2369	=	6139026.6
.8227	× 3000	× 2261	=	5580374.1
.7835	× 3000	× 2196	=	5161698.0
.7462	× 3000	× 2091	=	4680912.6
.7107	× 3000	× 1964	=	4187444.4
.6768	× 3000	× 1816	=	3687206.4
.6446	× 3000	× 1648	=	3186902.4
.6139	× 3000	× 1470	=	2707299.0
.5847	× 3000	× 1292	=	2266297.2
.5568	× 3000	× 1114	=	1860825.6
.5303	× 3000	× 933	=	1484309.7
.5051	× 3000	× 744	=	1127383.2
.4810	× 3000	× 555	=	800865.0
.4581	× 3000	× 385	=	529105.5
.4363	× 3000	× 246	=	322235.4
.4155	× 3000	× 137	=	170770.5
.3957	× 3000	× 58	=	68851.8
.3769	× 3000	× 18	=	20352.6
.3589	× 3000	× 3	=	3230.1
				$57674268.3

$57674268.3 ÷ 26237 = $2198.20, net premium.

(3.) 1st. Find the present worth of $1 for 1 *yr.*, 2 *yr.*, 3 *yr.*, etc., to 56 *yr.*, at compound interest, at 4%. Multiplying these by $2500 will give the present worth of $2500 for the respective times. These products are then to be multiplied by the fractions representing the contingency of death. The sum of these products will be the net single premium. In practice the division by the common denominator is omitted till the sum is taken, as explained above. The operation is as follows:

$.9615 × 2500 × 765 =
.9246 × 2500 × 774 =
.889 × 2500 × 785 =
.8548 × 2500 × 797 =

etc., etc., to 56 *yr.*, and the sum of the products divided by 78106.

(5.) Ex. 2 gives the net single premium in this case as $2198.209. Hence we are to find the total present worth of $1 paid annually, and by this amount divide the net annual premium. The work for finding the present worth of the payment of $1 *at the end of each year* is as follows:

.9524	×	23761	= 22629.9764
.9070	×	21330	= 19346.3100
.8638	×	18961	= 16378.5118
.8227	×	16670	= 13714.4090
.7835	×	14474	= 11340.3790
.7462	×	12383	= 9240.1946
.7107	×	10419	= 7404.7833
.6768	×	8603	= 5822.5104
.6446	×	6955	= 4483.1930
.6139	×	5485	= 3367.2415
.5847	×	4193	= 2451.6471
.5568	×	3079	= 1714.3872
.5303	×	2146	= 1138.0238
.5051	×	1402	= 708.1502
.4810	×	847	= 407.4070
.4581	×	462	= 211.6422
.4363	×	216	= 94.2408
.4155	×	79	= 32.8245
.3957	×	21	= 8.3097
.3769	×	3	= 1.1307

120495.2722 ÷ 26237 = $4.5925

Add $1 paid down, ... 1.0000

Total present worth of $1 annual premium, $5.5925

Hence \$2198.209 ÷ 5.5925 = \$393.06, net annual premium.

\$.9524 × $\frac{23761}{26237}$ gives present worth of \$1 for 1 *yr.* with the contingency of the person living 1 year; \$.9070 × $\frac{21330}{26237}$ gives the present worth of \$1 for 2 years subject to the contingency of the person living 2 years, etc., for 20 years. But instead of dividing each product by the common divisor, 26237, we divide the sum of the products by 26237.

NOTE.—Should the pupil be perplexed with the fact that in the solution of EX. 1, p. 259, we use the number of *deaths* as the numerator of the fractions, while in the case under EX. 4, p. 260, we use the number of *survivors*, all that will be necessary to relieve it is to call attention to the fact that in the former case the contingency grew out of the possibility of the person's *dying* during the time, while in the latter it grows out of the possibility of his *living*.

(6.) From EX. 2 in the text-book we have the *net single premium*, as before, \$2198.209. The following is the computation for finding the present worth of \$1 annual premium:

Present worth of \$1 paid down \$1.00

\$.9524	× 23761	= 22629.9764
.9070	× 21330	= 19346.3100
.8638	× 18961	= 16378.5118
.8227	× 16670	= 13714.4090
.7835	× 14474	= 11340.3790
.7462	× 12383	= 9240.1946
.7107	× 10419	= 7404.7833
.6768	× 8603	= 5822.5104
.6446	× 6955	= 4483.1930

110360.2675 ÷ 26237 = \$4.2063

Total present worth of \$1 for 10 *yr.* annual prem's, \$5.2063

\$2198.20 ÷ 5.21 = \$421.91, net annual premium.

(7.) For this purpose we should need a present worth table corresponding to (***348***), with a 4½% column, and extended to 61 *yr.* Such a table is computed according to the principle and formula developed under Ex. 11, p. 203. The question is what present sum will amount to $1 in 1 *yr.*, 2 *yr.*, 3 *yr.*, etc., at compound interest. The formula is $P_n = \frac{\$1}{(1+r)^n}$. Thus in 3 *yr.* at 4½% we find that $1 amounts to $(1.045)^3 = \$1.14114$. Hence to amount to $1, there will be required $\frac{\$1}{1.14114}$, or $0.8763. Having the proper table, we take out the several present worths corresponding to the various years from 1 to 61, and multiply each by the number of deaths as given in (***346***) corresponding to the respective years, beginning at 35, as the first year. The sum of these products divided by 81822 will give the *net single premium* to be paid to secure $1. This multiplied by 5000 will give the net single premium to secure a $5000 policy.

Second, find the present worth of $1 annuity for 61 *yr.* contingent on the life, the 1st payment being made down. This is done by adding to $1 the quotient of the sum of the 60 products obtained by multiplying the present worth of $1 for 1 *yr.*, 2 *yr.*, 3 *yr.*, etc., to 60 *yr.*, by the corresponding numbers of survivors taken from the table (***346***) beginning with 36 *yr.* (81090), divided by 81822, the number living at 35.

Finally, divide the net single premium by this present worth, and the quotient will be the net annual premium.

To find the net annual premium to secure a paid-up policy in 20 years, divide the net single premium by the quotient of the sum of the first 20 of the above present worths divided by 81822.

(10.) Same as above.

(11.) Office premium in Ex. 1, $1\frac{1}{4}$ times \$92.83, or \$111.40, at 20% for loading.

At 25%, $1\frac{1}{4}$ times \$92.83. At 40% $1\frac{1}{4}$ times \$92.83. Similarly in the other examples.

(14.) The present worth of \$1 *certain* 20 years hence is $\frac{\$1.00}{(1.05)^{20}} = .3769$. The probability of a person at 40 living to 60 is $\frac{77341}{78106}$. Hence the $P_{\cdot}$ of \$1 due 20 *yr.* hence and subject to this contingency is $\$0.3769 \times \frac{77341}{78106}$; and the $P_{\cdot}$ of \$15000 is \$15000 times this product. This gives the *single net premium.*

Second, find the present worth of \$1 premium for 20 *yr.* in the ordinary way, thus:

$$\begin{aligned} \$0.9524 \times 77341 &= \\ 0.9070 \times 76567 &= \\ 0.8638 \times 75782 &= \\ \text{etc., for 19 years.} & \end{aligned}$$

The sum of these products divided by 78106, and +\$1 will be the present worth sought. By this divide the net single premium and the quotient will be the net annual premium. This increased by the percentage charged as *loading* will be the *office annual premium.*

(16.) The solution consists in finding the *net single premium* as of a *simple endowment,* and then the *net single premium* as of *term endowment.* The sum of these is the *net single premium.* This net single premium divided by the sum of the present worths of \$1 premium for 30 *yr.* gives the *net annual premium.* The net annual premium multiplied by 1.30 gives the *office annual premium.*

These processes have been given in detail in the preceding articles, and should be reproduced here by the pupil.

This example affords an excellent opportunity to review the entire subject.

EXPECTATION OF LIFE.

(364.) PAGE 265.

This element, though of no special practical importance, the teacher may be curious to know how to compute.

The *Expectation of life* at any given age, is equal to the *sum* of the probabilities that the person will live over *one*, *two*, *three*, etc., years, increased by $\frac{1}{2}$.

To demonstrate this, let us find the expectation of life according to our Mortality Table of a person 90 years old. Now of the 847 persons alive at 90, 462 survive the year. Hence the person has on the average $\frac{462}{847}$ of a year for this year. Again of the 847 alive at 90, 216 survive to 92, thus giving $\frac{216}{847}$ of a year on the average to each individual for these two years. In like manner the probabilities for 3 *yr.* gives to each $\frac{79}{847}$ of a year; for 4 *yr.*, of $\frac{21}{847}$; for 5 *yr.*, of $\frac{3}{847}$; after this 0. Hence on these considerations, each of the 847 individuals would have an expectation of

$$\frac{462}{847} + \frac{216}{847} + \frac{79}{847} + \frac{21}{847} + \frac{3}{847} = \frac{781}{847}\ yr.$$

But this computation assumes that any life is equally liable to fail in any part of the year, and hence that the average liability is at the middle of the year. Thus, as the computation is based practically on the hypothesis that the life fails at the middle of the year, and the expectation includes the remainder of the year, we add $\frac{1}{2}$ to the above result, obtaining $\frac{781}{847} + \frac{1}{2} = 1.2$, very nearly.

It will be seen that, practically, all that is necessary in order to compute the expectation of life, as of a person 40 *yr.* old, from a mortality table like (*346*), is to add the numbers in the column of "survivors" from 41 through the table, double this sum and add the survivors at 40 (78106), and divide this sum by the last number (78106).

SIMPLE PARTNERSHIP.

PAGES 266, 267.

(2.) The entire stock is \$3500 + \$2500 + \$2500 + \$6500 = 15000. Now as each man's share of the gain is the same part of the whole gain that his stock is of the whole stock,

A has $\frac{3500}{15000} = \frac{7}{30}$ of the gain, or $\frac{7}{30}$ of \$4500 = \$1050.
B has $\frac{2500}{15000} = \frac{1}{6}$; $\frac{1}{6}$ of \$4500 = \$750.
C has $\frac{2500}{15000} = \frac{1}{6}$; $\frac{1}{6}$ of \$4500 = \$750.
D has $\frac{6500}{15000} = \frac{13}{30}$; $\frac{13}{30}$ of \$4500 = \$1950.

(3.) \$8280 + \$4968 = \$13248; \$24840 — \$13248 = \$11592.

$\frac{8280}{24840} = \frac{1}{3}$, P's part of expense; $\frac{1}{3}$ of \$9315 = \$3105.
$\frac{4968}{24840} = \frac{1}{5}$, Q's " " " $\frac{1}{5}$ of \$9315 = \$1863.
$\frac{11592}{24840} = \frac{7}{15}$, S's " " " $\frac{7}{15}$ of \$9315 = \$4347.

(4.) \$100 + \$200 + \$400 + \$700 = \$1400, whole gain.
$\frac{100}{1400} = \frac{1}{14}$; $\frac{200}{1400} = \frac{1}{7}$; $\frac{400}{1400} = \frac{2}{7}$; $\frac{700}{1400} = \frac{1}{2}$.
$\frac{1}{14}$ of \$2800 = \$200, A's capital.
$\frac{1}{7}$ of \$2800 = \$400, B's "
$\frac{2}{7}$ of \$2800 = \$800, C's "
$\frac{1}{2}$ of \$2800 = \$1400, D's "

(5.) \$4500 — \$600 = \$3900, to be divided.
$\frac{660}{11500} = \frac{13}{230}$; $\frac{13}{230}$ of \$3900 = \$22.04.
$\frac{400}{11500} = \frac{8}{230}$; $\frac{8}{230}$ of \$3900 = \$13.57.
$\frac{840}{11500} = \frac{84}{1150}$; $\frac{84}{1150}$ of \$3900 = \$28.79.

(6.) As it is assumed that I received the same part of my claim that other creditors did, the man paid $\frac{100}{250}$, or $\frac{2}{5}$ of his liabilities. $\frac{2}{5}$ of \$5500 = \$13750, total liabilities. $\frac{2}{5}$ of \$88 = \$35.20, the amount a man would receive whom A owed \$88.

COMPOUND PARTNERSHIP.

PAGES 267, 268.

(2.)

The use of \$300 for 5 mo. = the use of \$1500 1 mo. A's.
" " " \$400 " 8 mo. = " " " \$3200 1 mo. B's.
" " " \$500 " 3 mo. = " " " \$1500 1 mo. C's.

Use of entire capital for 1 mo. = the use of \$6200 1 mo.

$\frac{1500}{6200} = \frac{15}{62}$, A's part of gain. $\frac{15}{62}$ of \$100 = \$24.193.
$\frac{3200}{6200} = \frac{32}{62}$, B's " " " $\frac{32}{62}$ of \$100 = \$51.613.
$\frac{1500}{6200} = \frac{15}{62}$, C's " " " $\frac{15}{62}$ of \$100 = \$24.193.

(3.)

The pasture of 18 cows 3 mo. costs as much as 54 cows 1 mo.
" " " 24 " 4 mo. " " " " 96 " 1 mo.
" " " 12 " $3\frac{1}{2}$ mo. " " " " 42 " 1 mo.
The entire pasturage " " " "192 " 1 mo.

$\frac{54}{192}$ of \$38.40 = \$10.80.
$\frac{96}{192}$ of \$38.40 = \$19.20.
$\frac{42}{192}$ of \$38.40 = \$8.40.

(4.)

The first worked 250 *hr.* $\frac{250}{4102}$ of \$1200 = \$73.13.
" 2d " 880 *hr.* $\frac{880}{4102}$ of \$1200 = \$257.44.
" 3d " 972 *hr.* $\frac{972}{4102}$ of \$1200 = \$284.35.
" 4th " 2000 *hr.* $\frac{2000}{4102}$ of \$1200 = \$585.08.

Whole number of *hrs.*, 4102.

(5.)

\$2500 × 24 = \$60000
\$1000 × 16 = 16000
The use of A's capital = the use of \$76000 1 mo.
\$3000 × 9 = \$27000
\$2000 × 15 = \$30000
The use of B's capital = the use of \$57000 1 mo.
The use of the entire capital = the use of \$133000 1 mo.

Hence A's gain $= \frac{76}{133}$ of \$877.45, or \$501.40;
and B's gain $= \frac{57}{133}$ of \$877.45, or \$376.05.

(6.) The use of A's $\frac{1}{2}$ for $\frac{3}{4}$ *yr.* $= \frac{1}{2} \times \frac{3}{4} = \frac{3}{8}$ for 1 *yr.*
" " " B's $\frac{2}{5}$ " $\frac{1}{2}$ *yr.* $= \frac{2}{5} \times \frac{1}{2} = \frac{1}{5}$ for 1 *yr.*
" " " C's $\frac{1}{10}$ " 1 *yr.* $=$ $\frac{1}{10}$ for 1 *yr.*

Hence the partners are to receive of the \$5400 gain in the ratio of $\frac{3}{8}$, $\frac{1}{5}$, and $\frac{1}{10}$, or $\frac{15}{40}$, $\frac{8}{40}$, and $\frac{4}{40}$, or of 15, 8, and 4. Hence A has $\frac{15}{27}$ of \$5400 = \$3000 gain; to which adding $\frac{1}{2}$ of \$5400, his original stock gives him \$3000 + \$2700 = \$5700.

In like manner B has $\frac{8}{27}$ of \$5400 × \$2160 = \$3760, and C $\frac{4}{27}$ of \$5400 + \$546 = \$1340.

(7.) D's gain is $\frac{2}{5}$ of $\frac{2}{5}$, or $\frac{48}{720}$ in 1 *mo.*
B's " " $\frac{1}{4}$ of $\frac{1}{10}$, or $\frac{18}{720}$ in 1 *mo.*
C's " " $\frac{1}{8}$ of $\frac{1}{2}$, or $\frac{45}{720}$ in 1 *mo.*

Hence their shares of the gain are as 48, 18, and 45, and we have $\frac{48}{18} = \frac{\text{\$6000, D's}}{\text{\$2250, B's}}$, and $\frac{48}{45} = \frac{\text{\$6000, D's}}{\text{\$5625, C's}}$.

(8.)

$\frac{1}{10}$ of \$456 = \$45.60, A's; $\frac{456}{1510}$ of \$14345 = \$4332.
$\frac{1}{5}$ of \$343.20 = 42.90, B's; $\frac{422}{1510}$ of \$14345 = \$4075.50.
$\frac{1}{12}$ of \$750 = 62.50, C's; $\frac{625}{1510}$ of \$14345 = \$5937.50.
\$151.00

(9.) A's gain being \$1200 — \$960, or \$240 in 6 *mos.*, he gains \$40 in 1 *mo.*; and as his stock was \$960, his gain is $\frac{40}{960} = \frac{1}{24}$ of his stock each month, or $\frac{1}{2}$ his stock for the 12 *mos.* Now as B receives in the same ratio as A, \$2400 is $1\frac{1}{2}$ times his stock, and his stock was \$1600. Again C's gain is \$1040 — \$640 = \$400; hence his gain is $\frac{5}{8}$ of his stock. But in 1 *mo.* it would have been $\frac{1}{24}$ his stock. Hence $\frac{5}{8} \div \frac{1}{24} = 15$ is the number of months C's stock was in.

PROBLEMS FOR REVIEW.

(1.) $\frac{140 \times 80}{160} = 70.$

(2.) $\frac{(160)^2 \times 3.1416}{160} = 160 \times 3.1416 = 502.656.$

Circumference = 3.1416 times 1 *mi.* = 3.1416 *mi.*

(3.) 12 *hr.* — (9 *hr.* 1 *min.* 37 *sec.*) = 2 *hr.* 58 *min.* 23 *sec.* diff. in time. Multiplying by 15 we have diff. in long. 44° 35′ 45″. Adding this to long. of Rochester, we have 122° 26′ 45″, the long. of San Francisco.

(4.) (16 12.75) ÷ 4 = 4 3.1875. Hence the diff. in long. is 4° 3′.1875 = 4°.05311. 51.1 *mi.* × 4.05311 = 207.114 *mi.*

(5.) Since the earth makes a revolution on its axis in 24 hours = 1440 minutes, any point on its surface passes through 1° in $\frac{1}{360}$ of 1440 *m.*, or 4 *m.*; and in 1 hour it passes through $\frac{1}{24}$ of 360°, or 15°. Hence each degree = 4 minutes of time, and each hour 15 degrees of longitude.

(6.) $\frac{1}{24}$ of 360° = 15°. 360° ÷ 10.5 = 34° 28′.

75000 × 3.1416 = 235620, circumference.

235620 ÷ 24 = 9817.5 *mi.* = 1 *hr.* of longitude.

(7.)

83° 50′ 48″.3	89° 23′
71° 13′ 30″	83° 50′ 48″.3
12° 47′ 18″.3 Diff. longitude.	5° 32′ 11″.17 Diff. longitude.
4	4
51 *min.* 9 *sec*+ slow.	22 *min.* 9 *sec.*— fast.

(8.)

12 *hr.* 00	00	4) 44 *min.* 56½ *sec.*	Diff. in time.
59	43½	11° 14′ 7″.5	Diff. in long.
11 *hr.* 00	16½ *sec.*	83° 50′ 48″.3	Long. of A.A.
10 *hr.* 15 *min.*	20 *sec.*	95° 4′ 55″.8	Long. St. Paul.
44 *min.*	56½ *sec.*		

(9.) As there are 180° *Fah.* between freezing and boiling, and 100° *C.*, a degree *Fah.* $= \frac{100}{180}$, or $\frac{5}{9}$ of a degree *C.* Now 68° *Fah.* is 68° − 32° = 36° *Fah.* above freezing; or $\frac{5}{9}$ of 36° = 20° *C.*

(10.) $\frac{3}{4}$ of 432 = 324; $\sqrt{324} = 18$; 3 : 4 :: 18 : 24, the longest side; and $\frac{3}{4}$ of 24 = 18 is the shortest side.

(11.) $\frac{25}{4} \times \frac{7}{2} = \frac{175}{8}$ *sq. in.* $= \frac{175}{1152}$ *sq. ft.*
$\frac{25}{4}$ *cu. ft.* $\div \frac{175}{1152} = 41\frac{1}{7}$ *ft.*, in length.

(12.) 24 *hr.* = 1440 *min.* But as the watch loses 4 *min.* per day it indicates only 1436 *min.* Therefore the real time is $\frac{1440}{1436}$ times the indicated time. Hence when the watch indicates 10 *da.*, the true time is $\frac{1440}{1436}$ times 10 *da.*, or 10 *da.* 40 *min.* 7 *sec.*

(13.) It is sunrise to the people on the meridian 90° west longitude, because it is 6 hours before noon at the latter place; and it is sunset to the people on the meridian 90° east longitude, because it is 6 hours after noon on this meridian. It is midnight on the 180th meridian.

(14.) 156 − 144 = 12 *ft.* gain per minute; 864 ÷ 2 = 432 to be gained. 432 ÷ 12 = 36 *min.* Since B gains 12 *ft.* per minute, it will take as many minutes to gain 432 *ft.* as 12 is contained times in 432, or 36 *m.*

144 × 36 = 5184 *ft.*, distance A travels.
156 × 36 = 5616 " " B "

(15.) 12 × 40 × 10 = $4800 *cu. ft.*; 4800 ÷ 128 = 37½ *cords.* $4\frac{1}{2} \times 37\frac{1}{2} = \$159.37\frac{1}{2}$.

(16.) $9 \times 46 = 414$ *sq. ft.*; $414 \div 15.75 = 26\frac{4}{7}$ *ft.* in length.

(17.) $18 \times 35 = 630$ *sq. ft.* $630 \div 9 = 70$ *sq. yds.*
$70 \div \frac{7}{8} = 70 \times \frac{8}{7} = 80$ *yds.*; $\$1.75 \times 80 = \$140$, cost.

(18.) $180° - 82°\ 58' = 97°\ 2'$ E. longitude, midnight.
$90° - 82°\ 58' = 7°\ 2'$ " 6 A. M.
$90° + 82°\ 58' = 172°\ 58'$ W. " 6 P. M.
180° east of Detroit is 97° 2′ east longitude.
90° west " " " 172° 58′ west "

(19.) $8 \times 4 \times 2 = 64$ *cu. in.* in one brick; $64 \times \frac{5}{4} = 80$ *cu. in.*, what each brick and its mortar will measure in the wall.

$231 \times 2000 = 462000$ *cu. in.* inside of cistern.

$\sqrt[3]{462000} = 77.3$, number of *in.* inside measure; and 77.3 *in.* $+ 24$ *in.* $=$ the outside measure.

$101.3 \times 101.3 \times 12 = 123140.28$ *cu. in.* bottom.
$77.3 \times 89.3 \times 12 \times 4 = 331338.72$ *cu. in.* in 4 sides.
454479 *cu. in.* in whole wall.
$454479 \div 80 = 5681$ bricks.

(20.)

$16 \times 3\frac{3}{4} = 60$
$19 \times 8 = 152$
$19 \times 6\frac{3}{4} = 128\frac{1}{4}$
$340\frac{1}{4}$ *rd.*

$340\frac{1}{4} \div 2 = 170\frac{1}{8}$ *sq. rd.* $= 1$ *A.* $10\frac{1}{8}$ *rd.*

(21.) The hands of a watch are together at 12 o'clock; and the minute-hand gains on the hour-hand at the rate of 11 of the hour spaces in 1 hour. Again, for the hands to be opposite and the hour-hand between 9 and 10, the minute-hand must gain $8\frac{1}{2}$ revolutions, or $8\frac{1}{2}$ times 12 $= 102$ spaces. This will require $\frac{102}{11} = 9\frac{3}{11}$ hours. Now from 12 M. of Mar. 24th to $9\frac{3}{11}$ hours past midnight

in the morning of Mar. 30, is 5 *da.* $21\frac{3}{11}$ *hr.*, or $5\frac{11}{132}$ *hr.* But this is indicated time, which, as the watch gains 2 *min.* per day, is to the true time as $\frac{1442}{1440} = \frac{721}{720}$. Hence letting x represent the true time, we have $\frac{721}{720} = \frac{5\frac{11}{132}}{x}$, or $\frac{721}{720} = \frac{777}{132x}$; whence $x = \frac{111 \times 60}{11 \times 103} = 5$ *da.* 21 *hr.* 4 *min.* $36\frac{5}{103}$ *sec.* Hence the time of day is 9 o'clock, 4 *min.* 36.4 *sec.* +.

(22.) $\sqrt{(11)^2 + 5^2} = \sqrt{146} = 12.083$, or 12 *ft.* .996 *in.*

(23.) $16\frac{1}{2}$ *ft.* $\times 5 = 82.5$ *ft.* $\sqrt{(82.5)^2 + (60)^2} = \sqrt{10406.25} = 102+$. 102 *ft.* $= \frac{102}{16\frac{1}{2}}$ *rd.* $= 6$ *rd.* 3 *ft.*

(24.) $7 \times 7 \times .7854 \times 8 = 307.8768$, the No. *cu. ft.*
$307.8768 \times \frac{1728}{231} = 2302.965$, the No. *gal.*
$2302.965 \div 31\frac{1}{2} = 73.11$, the No. *bbls.*

(25.) $4 \times (3960)^2 \times 3.1416 = 197{,}061{,}258.24$, the No. *sq. mi.* in the surface.

(26.) Surface $= 4 \times (\frac{1}{2})^2 \times 3.1416$ *sq. ft.* $= 3.1416$ *sq. ft.*
Volume $= \frac{1}{6}$ of 3.1416 *cu. ft.* $= .5236$ *cu. ft.*
Vol. earth $= \frac{1}{6} \times 3960 \times 4 \times (3960)^2 \times 3.1416$ *cu. mi.* $= 260{,}120{,}860{,}876.8$ *cu. mi.*

(27.) First draw a line between the middle points of the two ends, as AB in (***368***). Then take a cord 10 *ft.* long and fastening its center at the middle of one of the sides, as at C in the cut, fasten the extremities in the line AB, as at F′ and F in the cut. Then with a sharpened stick placed against the cord carry it around, as represented in the cut. *The area would be* $3.1416 \times 5 \times 3$, or 47.124 *sq. ft.*

(28.) *Area* $= \frac{3.1416 \times 50 \times 30}{160}$ *A.* $= 29.4525$ *A.*
If the axes of an ellipse are equal the figure becomes a

circle with the foci at the center. The formula πab, which means 3.1416 multiplied into the product of the semi-axes, becomes πr^2, r being one of the semi-axes, or the radius of the circle, and is the common formula for the area of a circle.

(29.) $3.1416 \times 6^2 \times \frac{1}{3}$ *cu. ft.* $= 150.8$ *cu. ft.*, nearly.

(30.) $3.1416 \times 2^2 \times \frac{1}{3}$ of $5\frac{1}{2} = 22.3402$, number of *cu. ft.* in smaller cone.

$3.1416 \times 5^2 \times \frac{1}{3}$ of $13\frac{1}{2} = 349.0666$, number of *cu. ft.* in larger cone.

$349.0666 - 22.3402 = 326.7264$, number of *cu. ft.* in frustum.

$\frac{326.7264 \times 1728}{231 \times 31\frac{1}{2}} = 77.59$, number of *bbl.* contained.

The whole operation may be indicated as follows:

$$\frac{\frac{1}{3} \text{ of } 3.1416(5^2 \times 13\frac{1}{2} - 2^2 \times 5\frac{1}{2}) \times 1728}{231 \times 31\frac{1}{2}} = 77.59.$$

(31.) $\sqrt{160 \times 10} = 40$, number of *rd.* on a side $=$ 660 *ft.* Entire length of walk $4(660 + 4) = 2656$ *ft.* Area $= 2656 \times 4 = 10624$ *sq. ft.* Cost of paving $=$ $10624 \times 6\frac{1}{4}c. = \$664$.

(32.) Using the cut for Ex. 30 in the text-book, ABCD represents an inverted section of the coffee-pot, AB $= 7$ *in.*, the diameter of the bottom, CD $= 4\frac{1}{2}$ *in.*, the diameter of the top, and GE $= 8\frac{1}{2}$ *in.*, the depth.

To find the entire altitude of the completed cone we have

HB$(= \frac{1}{2})$: EB$(= \frac{1}{4})$ :: HD$(=$ EG $= 8\frac{1}{2})$: EF, or 5 : 14 :: $8\frac{1}{2}$: EF $= 23.8$. Hence FC, the altitude of the smaller cone, is $23.8 - 8.5 = 15.3$.

Then as above $\frac{\frac{1}{3} \text{ of } 3.1416[(\frac{7}{2})^2 \times 23.8 - (\frac{3}{2})^2 \times 15.3]}{57.75} =$ 3.88 *qt.*, or very nearly 1 *gal.*

(33.) Slant height of larger cone $= \sqrt{(13\frac{1}{3})^2 + 5^2} =$ 14.24 *ft.*; and of smaller, $\sqrt{(5\frac{1}{3})^2 + 2^2} = 5.7$ *ft.* Area of convex surface of larger cone $= \frac{1}{2}(2 \times 3.1416 \times 5 \times$ 14.24), and of smaller, $\frac{1}{2}(2 \times 3.1416 \times 2 \times 5.7)$; hence the convex surface of the frustum is

$\frac{1}{2} \times 2 \times 3.1416(5 \times 14.24 - 2 \times 5.7) = 187.8677$
Area of bottom, $3.1416 \times 2^2 = 12.5664$
Total area, 200.43+.

(34.) $20 - 17 = 3$; $17 + 2 = 19$;
$19 \times 19 \times .7854 \times 28 = 7938.8232$, number of *cu. in.*
$7938.8232 \div 231 = 34.37$, number of *gal.*
$\frac{2}{3}$ of $(23 - 19) = 2\frac{2}{3}$; $(19 + 2\frac{2}{3})^2 \times .7854 \times 31 \times \frac{1}{231}$ $= 49.48+$.

(35.) $(18)^2 = 324$; $324 + 324 = 648$; $\sqrt{648} = 25.45$, number of *in.* on a side.

(36.) $(15)^2 = 225$; $225 \div 2 = 112.5$; $\sqrt{112.5} = 10.6$ $= \frac{1}{2}$ diameter.

(37.) If the diameter is 1 the radius is $\frac{1}{2}$; $(\frac{1}{2})^2 = \frac{1}{4}$.
$\frac{1}{4} + \frac{1}{4} = \frac{1}{2}$; $\sqrt{\frac{1}{2}} = \sqrt{\frac{2}{4}} = \frac{1.4142}{2} = .7071$.

(38.) 16 *in.* $= 1\frac{1}{3}$ *ft.*; $1\frac{1}{3} \times 1\frac{1}{3} \times 24 = 42\frac{2}{3}$, number of *cu. ft.*

(39.) Let r represent the radius; then $\sqrt{r^2 + r^2} = r\sqrt{2}$ is the side of the inscribed square. (See cut, Ex. 35.) Now the arc of the square is $(r\sqrt{2})^2 = 2r^2$.

(40.) A diagonal passing through the cube is the diameter of the sphere. But such a diagonal is the hypotenuse of a right-angled triangle, the sides of which are an edge of the cube and the diagonal of one of its faces. The square of the latter diagonal being twice the square of an edge, the square of the diagonal of the cube is 3 times the

square of an edge. Hence $\sqrt{(40)^2} = \sqrt{533\frac{1}{3}} = 23.08$, is the edge.

(41.) If radius $= 5\ ft.$, diameter $= 10\ ft.$

$3.1416 \times 10 = 31.416$, circumference of the circle.

$\frac{75}{360} = \frac{5}{24}$; $\frac{5}{24}$ of $31.416 = 6.545$, length of arc.

(42.) Diameter $= 20\ ft.$; $20 \times 3.1416 = 62.832$, circ. of circle.

$\frac{3.000}{62.832}$ of $360° = 17° 11' 19''.3$.

(43.) Cross section of pipe $= 1^2 \times 3.1416 = 3.1416$ *sq. in.*, or $3.14\frac{16}{144}$ *sq. ft.* 10 *hr.* $= 600$ *min.* Hence the discharge is 20×600, or 12000 *ft.* per hour. The volume discharged is therefore $\frac{3.1416 \times 12000 \times 1728}{144 \times 231 \times 31\frac{1}{2}} = 62.17$ *bbl.*

(44.)

$$\frac{2.75 + 2.7 + 4.1 + 4.8 + 4.8 + 4.05 + 4.12 + 8 + \frac{1}{2}(8.9 + 4)}{10 - 1}$$

$= \frac{41.77}{9}$, the mean width in chains.

Hence $\frac{41.77 \times 19}{9 \times 90} = 8.81\frac{81}{90}$, the number of acres.

(45.) Average width $\frac{359 + 351}{2} = 355\ ft.$

Area of cross section, $355 \times 5\frac{1}{2} = 1952\frac{1}{2}$ *sq. ft.*

Rate of current, $2 \times 24 \times 5280 = 253440\ ft.$ per day.

Hence the volume discharged $= \frac{1952\frac{1}{2} \times 253440 \times 1728}{231 \times 31\frac{1}{2}}$ $= 117513404$ *bbl.*

(46.) There being 8 measurements, including extremes of depths, and the extremes being 0, the *average* depth $= \frac{2 + 3 + 4 + 4\frac{1}{2} + 2 + 1\frac{1}{2}}{8 - 1} = \frac{17}{7} = 2\frac{3}{7}\ ft.$ In like manner, there being 7 measurements of rate of current, including

shore rates, which are 0, we have the *average rate*, $\frac{1056 + 2640 + 5280 + 1320 + 528}{7 - 1} = 1804$ *ft.* per hour.

Hence the volume discharged per day =

$$\frac{560 \times 2\frac{3}{4} \times 1804 \times 1728 \times 24}{231 \times 31\frac{1}{2}} = 13983242.4 \text{ bbl.}$$

DOYLE'S RULE.

PAGE 277.

(2.) $\frac{1}{8}$ of $32 \times 32 = 1152$: $\frac{11}{16}$ of $32 \times 32 = 960$: $\frac{5}{6}$ of $32 \times 32 = 1280$.

(3.) $\frac{4}{5}$ of $40 \times 40 = 1000$: $\frac{3}{5}$ of $40 \times 40 = 2400$.

(4.) $\frac{3}{4}$ of $8 \times 8 = 48$: $\frac{9}{8}$ of $8 \times 8 = 72$: $8 \times 8 = 64$.

(5.) $11 \times 11 = 121$: $\frac{11}{16}$ of $11 \times 11 = 83+$: $\frac{9}{8}$ of $11 \times 11 = 136+$: $1\frac{1}{8}$ of $11 \times 11 = 227-$.

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